Calculus : Day 1

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1. Sequences

A sequence of real numbers means writing some real numbers one after another, like $1, \frac{1}{2}, \frac{1}{3}, \cdots$ or $1, -1, 1, -1, \cdots$ etc. When we say 'sequence', it is by default that there are infinitely many[1] of them. When we have finitely many numbers, we call it as a 'finite sequence'. The formal definition of a sequence is given below.

Definition (Sequence) A sequence of real numbers is a function $f: \mathbb{N} \to \mathbb{R}$, written as $f(1), f(2), f(3), \cdots$ or as $a_1, a_2, \cdots$ where $a_n = f(n)$, $n \geq 1$.

You might be thinking "What is the difference between a sequence and an infinite set?". The difference is that, as a set, $\{-1, 1, -1, 1, \ldots\}$ is same as $\{-1, 1\}$. But as a sequence, $(-1, 1, -1, 1, \cdots)$ is not same as $(-1, 1)$ (infact, the later is not a sequence!).

Another thing to notice is that, there need not be any closed-form expression for a_n. Just as in the case of functions, a_n's can be defined without having a closed-form expression. For instance, $a_n = n^{th}$ prime number, $n \geq 1$, does not have a closed-form formula for a_n.

Notation The sequence $a_1, a_2, \cdots$ is denoted by $\{a_n\}_{n\geq 1}$, $(a_n)_{n\geq 1}$ or $\{a_n\}_{n=1}^{\infty}$ etc. Note, the index need not start from 1: $(n^2)_{n=0}^{\infty}$ or $\left\{\frac{1}{n-1}\right\}_{n=2}^{\infty}$ are also sequences.

Next, we shall define some concepts related to a sequence. We say that $(x_n)_{n\geq 1}$ is an increasing sequence if $x_{n+1} \geq x_n$ holds for every $n \geq 1$. Similarly, we say

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1 Countably infinite, to be precise.

~~we~~ that $(x_n)_{n\geq 1}$ is decreasing if $x_{n+1} \leq x_n$ holds for every $n \geq 1$. If a sequence is either increasing or decreasing, we say that it is a monotonic sequence.

We say that $\{x_n\}_{n\geq 1}$ is bounded from above if ~~x~~ there exists a real number B such that $x_n \leq B$ holds for every $n \geq 1$. Similarly, $\{x_n\}_{n\geq 1}$ is called bounded from below if there exists a real number A such that $A \leq x_n$ for every $n \geq 1$. We say that $\{x_n\}_{n\geq 1}$ is bounded if it is both bounded from above ~~or~~ and bounded from below. Thus, $\{x_n\}_{n\geq 1}$ is bounded if there exists $A, B \in \mathbb{R}$ such that $A \leq x_n \leq B$ for all $n \geq 1$. Note, here A, B need not be positive.

Exercises

1.1. Can a sequence be both increasing and decreasing?

1.2. Show that $\{a_n\}_{n\geq 1}$ is increasing if and only if $\{-a_n\}_{n\geq 1}$ is decreasing.

1.3. Show that $\{a_n\}_{n\geq 1}$ is bounded from above if and only if $\{-a_n\}_{n\geq 1}$ is bounded from below.

1.4. Consider the sequences $\{\sin \frac{\pi}{2n}\}_{n\geq 1}$ and $\{\cos \frac{\pi}{2n}\}_{n\geq 1}$. Are they increasing or decreasing?

1.5. Consider the sequence $\{2^n - n^2\}_{n\geq 1}$. Is it monotonic?

1.6. Find (with proof) which of the following sequences are bounded:- (a) $\{\sin n\}_{n\geq 1}$ (b) $\left\{\frac{2^{-n}}{n}\right\}_{n\geq 1}$ (c) $\left\{\frac{2^n}{n}\right\}_{n\geq 1}$

(d) $\left\{\sqrt{n^2+1} - n\right\}_{n\geq 1}$. ~~(e)~~ $\cancel{\left\{\frac{n}{\sin n}\right\}_{n\geq 1}}$

1.7. Let $\{a_n\}_{n\geq 1}$ be a sequence. Define another sequence $\{s_n\}_{n\geq 1}$ as $s_n = (a_1 + \dots + a_n)/n$, $n \geq 1$.

(a) If $\{a_n\}_{n\geq 1}$ is bounded, show that $\{s_n\}_{n\geq 1}$ is also bounded.

(b) If $\{s_n\}_{n\geq 1}$ is bounded, is it necessary that $\{a_n\}_{n\geq 1}$ must be bounded?

1.8. Show that $\{x_n\}_{n\geq 1}$ is bounded if and only if there exists $M > 0$ such that $|x_n| \leq M$ holds for all $n \geq 1$.

1.9. Determine, with proof, whether the following sequences are bounded:-

(a) $x_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)}, \quad n \geq 1.$

(b) $x_n = \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}+\sqrt{n}}, \quad n \geq 1.$

(c) $x_n = \frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots$ (upto n^{th} term), $n \geq 1$.

(d) $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}, \quad n \geq 1.$

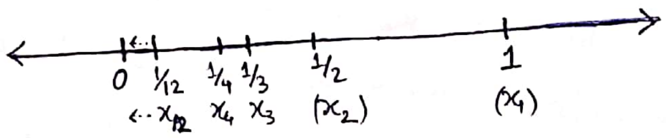
(e) $x_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}, \quad n \geq 1.$

(f) $x_n = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)}, \quad n \geq 1.$

1.10. Let $x_n = \frac{n}{\sin n}$, $n \geq 1$. Is $\{x_n\}_{n\geq 1}$ bounded?

2. Limit of a Sequence

Consider the sequence $\{\frac{1}{n}\}_{n\geq 1}$. The sequence is a decreasing sequence. It is bounded because $0 < \frac{1}{n} \leq 1$, for every $n \geq 1$. What else can you tell about this sequence?



If we observe the terms on the number line, we see that they are getting closer and closer to the number 0. We say that the sequence $x_n = \frac{1}{n}$ approaches 0 as n tends to infinity. We write it as $\lim_{n\to\infty} \frac{1}{n} = 0$.

Formally, we define limit of a sequence as follows —

Definition (Limit of a sequence)

We say that "x_n converges to x as $n \to \infty$" if, given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, x_n belongs to $(x - \varepsilon, x + \varepsilon)$. (i.e. $|x_n - x| < \varepsilon$.)

When x_n converges to x as $n \to \infty$, ~~we also say it~~ we write "$x_n \to x$ as $n \to \infty$" or "$\lim_{n\to\infty} x_n = x$".

Example: $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof Take any $\varepsilon > 0$. There exist a natural number N which is greater than $1/\varepsilon$. (why?) Now, $N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon$.

And thus, for every $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore, $\lim_{n\to\infty} \frac{1}{n} = 0$. (from definition) ∎

Example: $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Proof The proof is essentially same as above, because

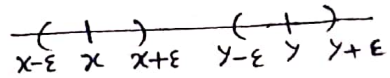
$$\left|\frac{(-1)^n}{n} - 0\right| = \left|\frac{1}{n} - 0\right|, \text{ for all } n \in \mathbb{N}.$$ ∎

Let us now see some properties of ~~a~~ limit of a sequence.

1. Limit of a sequence is unique, if it exists.

Proof. Let, if possible, $\{x_n\}_{n\geq 1}$ be a sequence such that it converges to both x and y, where $x \neq y$. Say, $x < y$. Intuitively we understand that it is not possible for the terms to get closer and closer to both x and y, because x, y are different.

To prove it rigorously, let us take $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ and $(y - \varepsilon, y + \varepsilon)$ are disjoint. Note, since we assumed $x < y$, it is enough to take $\varepsilon > 0$ such that $x + \varepsilon < y - \varepsilon$ or, $\varepsilon < \frac{y - x}{2}$. So, we can just take ε to be $\frac{y-x}{3}$ and then we get a contradiction (because x_n can't belong to ~~these~~ two disjoint sets at the same time!). ∎



$[(x - \varepsilon, x + \varepsilon)$ and $(y - \varepsilon, y + \varepsilon)]$

$\boxed{\text{2. If } \{x_n\}_{n\geq 1} \text{ converges then it must be bounded.}}$

~~Proof~~ Let us first understand why this should hold. Suppose $\{x_n\}_{n\geq 1}$ converges to x. Then, after a cut-off for the index n, all the ~~rest~~ x_n's ~~after~~ belong to $(x-\varepsilon, x+\varepsilon)$. So except finitely many terms, all the other terms are bounded inside $(x-\varepsilon, x+\varepsilon)$. Since only finitely many terms are left, we are able to bound the whole sequence. ~~..~~

Proof. Lets fix $\varepsilon = 1$. There exists $N \in \mathbb{N}$ such that for every $n \geq N$, $|x_n - x| \leq 1$ holds. Then, for every $n \geq N$, we get

$$|x_n| - |x| \leq |x_n - x| < 1 \Rightarrow |x_n| \leq |x| + 1.$$

~~Then take~~ Now, take $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x|+1\}$.

Clearly, $|x_n| \leq M$ holds for all $n \geq 1$. Hence $\{x_n\}_{n\geq 1}$ is bounded. ■

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3. Suppose $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$. And $c \in \mathbb{R}$. Then,

(a) $\lim_{n\to\infty} c x_n = cx$. (b) $\lim_{n\to\infty} (x_n + y_n) = (x+y)$.

(c) $\lim_{n\to\infty} x_n y_n = xy$. ~~(d)~~

Proof. (a) Let $c \neq 0$. Since $x_n \to x$ as $n \to \infty$, so for $\varepsilon' = \varepsilon/|c|$ there exists $N \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{|c|}$ ~~for~~ holds for every $n \geq N$. Hence, $|c x_n - cx| < \varepsilon$ holds for every $n \geq N$.

Thus, for every $\varepsilon > 0$, we get $N \in \mathbb{N}$ for which $|cx_n - cx| < \varepsilon$ holds for all $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} cx_n = cx$, from definition.

Note, if $c = 0$ then the result is trivial.

(b) idea: $|(x_n+y_n) - (x+y)| \leq \underbrace{|x_n - x|}_{<\varepsilon/2 \text{ for all } n\geq N_1} + \underbrace{|y_n - y|}_{<\varepsilon/2 \text{ for all } n \geq N_2} < \varepsilon/2 + \varepsilon/2 = \varepsilon$, for all $n \geq N_1, n \geq N_2$.

~~Pr~~ Take any $\varepsilon > 0$.

Since $x_n \to x$, so there exists N_1 s.t. $|x_n - x| < \varepsilon/2$ for all $n \geq N_1$.

Again, $y_n \to y$, so there exists N_2 s.t. $|y_n - y| < \frac{\varepsilon}{2}$ for all $n \geq N_2$.

Then, for every $n \geq N = \max\{N_1, N_2\}$, it holds that

$$\begin{aligned}|x_n + y_n - (x+y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \quad \text{(triangle inequality)} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad (\because n \geq N_1 \text{ and } n \geq N_2)\end{aligned}$$

Therefore from definition, $\lim_{n\to\infty} (x_n + y_n) = (x+y)$.

(Note, we took $N = \max\{N_1, N_2\}$ just to ensure that $n \geq N$ implies $n \geq N_1$ and $n \geq N_2$. We could take $N = N_1 + N_2$ or $N_1 N_2$ also!)

(c) idea: $$\begin{aligned}|x_n y_n - xy| &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \\ &\leq \underbrace{M \cdot |y_n - y|}_{<\varepsilon/2} + \underbrace{|y| \cdot |x_n - x|}_{<\varepsilon/2} \quad (\because x_n \text{ converges, it must be bounded})\end{aligned}$$
(for sufficiently large n)

Can you now complete the proof yourself? ∎

Question: Suppose x_n converges to x as $n \to \infty$. If $x_n \neq 0$ for all $n \geq 1$, then is it necessary that $x \neq 0$?

Answer: No. Take $x_n = \frac{1}{n}$, as a counter-example.

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4. Suppose $\{x_n\}_{n\geq 1}$ is a sequence that converges to x. Assume that $x_n \neq 0$ for all $n \geq 1$ and $x \neq 0$. Then, $\{1/x_n\}_{n\geq 1}$ converges to $1/x$.

Idea: $\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{|x_n| \cdot |x|} \overset{?}{<} \frac{|x_n - x|}{m\,|x|} < \varepsilon$ for all sufficiently large n.

We need an $m > 0$ such that $|x_n| > m$ holds for all sufficiently large n.

How to get such an m? Hint: Go back to the proof of #2 (proof of convergent $\Rightarrow$ bounded)

Proof. ~~Take~~ Fix $\varepsilon > 0$ (will be decided later). There exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

Now, $|x| - |x_n| \leq |x_n - x| < \varepsilon \Rightarrow |x_n| > |x| - \varepsilon$, for all $n \geq N$.

Now, we want to take ε such that $|x| - \varepsilon > 0$. So we can just take $\varepsilon = \frac{|x|}{2}$. Then, we have, ~~for all n ≥ N, that~~

$$|x_n| > |x| - \varepsilon = \frac{|x|}{2} > 0, \text{ for all } n \geq N. \quad \text{——— (I)}$$

Again, since x_n converges to x, taking $\varepsilon' = \varepsilon \frac{|x|^2}{2} > 0$, we have an $N' \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \frac{|x|^2}{2}, \text{ for all } n \geq N', \quad \text{——— (II)}$$

Using (I) and (II), we get for every $n \geq \max\{N, N'\}$,

$$\left| \frac{1}{x_n} - \frac{1}{x} \right| = \frac{|x_n - x|}{|x_n| \cdot |x|} < \frac{|x_n - x|}{|x|^2/2} < \varepsilon .$$

This completes the proof that $\left\{\frac{1}{x_n}\right\}_{n \geq 1}$ converges to $\frac{1}{x}$. ■

Corollary: If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and $y_n \neq 0$ for all $n \geq 1$, and $y \neq 0$, then $\frac{x_n}{y_n} \to \frac{x}{y}$ as $n \to \infty$.

Exercises

1.11 Show that (a) $\lim_{n \to \infty} \frac{1}{n^2} = 0$, (b) $\lim_{n \to \infty} \frac{n}{2n+3} = \frac{1}{2}$.

1.12 Suppose x_n converges to x and y_n converges to y. Show that $x_n - y_n$ converges to $x - y$.

1.13 Suppose $\{x_n\}_{n \geq 1}$ is a sequence of non-negative real numbers. If $\lim_{n \to \infty} x_n = x$, then show that $x \geq 0$.

1.14 Suppose $\{x_n\}_{n \geq 1}$ is a sequence such that $x_n \geq 0$ for every $n \geq 1$ and suppose that x_n converges to x. Then show that, $\sqrt{x_n}$ converges to $\sqrt{x}$.

1.15 Suppose $\lim_{n \to \infty} x_n = x$. Show that $\lim_{n \to \infty} x_n^k = x^k$, for any positive integer k. Hence show that $\lim_{n \to \infty} P(x_n) = P(x)$ holds for any polynomial $P(t)$ with real coefficients.

1.16 Let $x_n = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ 1 & \text{if } n \text{ is even}. \end{cases}$ Show that x_n does not converge to 1.

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1.17. Define $x_n = (-1)^n$. Show that x_n does not converge to any $x \in \mathbb{R}$. (We say that x_n diverges.)

1.18. Define $x_n = 2^n$. Show that x_n does not converge to any $x \in \mathbb{R}$. (We say that x_n diverges to $+\infty$. Some people still write that $\lim_{n\to\infty} x_n = +\infty$, in this sense.)

1.19. Suppose that $\lim_{n\to\infty} (x_n + y_n)$ exists. Does it imply that $\lim_{n\to\infty} x_n$ or $\lim_{n\to\infty} y_n$ exists? Does the conclusion change if we restrict x_n, y_n to be sequences of positive numbers?

1.20. Suppose that $\lim_{n\to\infty} (x_n + y_n)$, $\lim_{n\to\infty} (y_n + z_n)$, $\lim_{n\to\infty} (z_n + x_n)$ exists. Is it necessary that $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} z_n$ must exist?

1.21. (a) If $x_n \to x$ as $n \to \infty$, then show that $|x_n| \to |x|$ as $n \to \infty$.

(b) If $|x_n| \to |x|$ as $n \to \infty$, does it imply that $x_n \to x$ as $n \to \infty$?

(c) If $|x_n| \to 0$ as $n \to \infty$, does it imply that $x_n \to 0$ as $n \to \infty$?

1.22. Suppose that $\{x_n\}_{n\geq 1}$ converges but $\{y_n\}_{n\geq 1}$ does not converge. Is it possible that $\lim_{n\to\infty} (x_n - y_n)$ exists?

1.23. We saw that convergent sequences must be bounded. Is the converse true?

1.24. Suppose $x_n \to x$ as $n \to \infty$.

(a) If $x_n \geq a$ for all $n \geq 1$, then show that $x \geq a$.

(b) If $x_n \leq b$ for all $n \geq 1$, then show that $x \leq b$.

1.25. Can we make a convergent sequence ~~to a~~ divergent by changing finitely many terms? Can we make a divergent sequence convergent by changing finitely many terms?