Sandwich Theorem

Recall that we showed the following fact as an exercise: If $x_n \geq 0$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = x$ then we must have $x \geq 0$. Following are some consequences of this fact:

(a) If $x_n \leq 0$ for all $n \geq 1$, ~~then~~ and $\lim_{n\to\infty} x_n = x$ then $x \leq 0$.

(b) If $a \leq x_n \leq b$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = x$ then $a \leq x \leq b$.

(c) If $x_n \leq y_n$ for all $n \geq 1$ then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$ provided these limits exist.

Note, if we have strict inequality for x_n, the corresponding inequality for $\lim_{n\to\infty} x_n$ ~~does~~ need not remain strict. For example, if $x_n > 0$ for all $n \geq 1$, and $\lim_{n\to\infty} x_n = x$, it need not be the case that $x > 0$. (Take $x_n = \frac{1}{n}$.)

Now, suppose we have $y_n \leq x_n \leq z_n$ for all $n \geq 1$. And suppose we know that each of them converges as $n \to \infty$. If $\lim_{n\to\infty} y_n = l = \lim_{n\to\infty} z_n$, then we are able to conclude that $\lim_{n\to\infty} x_n = l$. Are we able to conclude this even without knowing that x_n converges? The answer turns out to be 'Yes', ~~And in fact~~ as given in the following theorem.

Theorem (Sandwich)

Suppose that $y_n \le x_n \le z_n$ for all $n \ge 1$ and $\lim_{n\to\infty} y_n = l = \lim_{n\to\infty} z_n$. Then $\lim_{n\to\infty} x_n$ exists and equals l.

~~Proof~~. [Suppose you are eating a sandwitch. The sequence z_n is in the upper loaf of bread and y_n is in the lower loaf of bread and x_n lies ~~betw~~ in the middle. And your mouth is the interval $(l-\varepsilon, l+\varepsilon)$. Since y_n and z_n are going inside your mouth, x_n being ~~sandwiched~~ 'squeezed' between y_n and z_n, must go inside your ~~moth~~ mouth $(l-\varepsilon, l+\varepsilon)$ too!]

Formal Proof: Pick any $\varepsilon > 0$. $\exists\, N_1$ such that for all $n \ge N_1$,

$$l-\varepsilon < y_n < l+\varepsilon \text{ holds.}$$

Similarly, $\exists\, N_2$ such that for all $n \ge N_2$,

$$l-\varepsilon < z_n < l+\varepsilon \text{ holds.}$$

Hence, for every $n \ge N = \max\{N_1, N_2\}$, we have

$$l-\varepsilon < y_n \le x_n \le z_n < l+\varepsilon.$$

This completes the proof. $\square$

Applications

There are many many applications of Sandwich theorem (also called squeezing principle). Some of the important applications are given below.

Example 1. Show that $\lim_{n\to\infty} 2^{-n} = 0$.

Of course you can do with from definition (for every $\varepsilon > 0$, show the existence of $N \in \mathbb{N}$ such that $|2^{-n} - 0| < \varepsilon$ holds for all $n \geq N$.) but it requires ~~little~~ some computation for N. Here is how we can do it, using Sandwich!

Soln We know that $2^n \geq n$ holds for all $n \in \mathbb{N}$ (can be proved by induction). Hence $0 < \frac{1}{2^n} < \frac{1}{n}$ for all $n \geq 1$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, Sandwich theorem applies and tells us that $\lim_{n\to\infty} 2^{-n}$ exists and equals zero. $\square$

Example 2. Show that $\lim_{n\to\infty}\left(\sqrt{n^2+1} - n\right) = 0$.

Soln: $\sqrt{n^2+1} - n = \dfrac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \dfrac{1}{\sqrt{n^2+1}+n}$

So, $0 < \sqrt{n^2+1} - n = \dfrac{1}{\sqrt{n^2+1}+n} < \dfrac{1}{n}$ for all $n \geq 1$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, Sandwich completes the proof. $\square$

Example 3. Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

~~Solution~~ Soln: $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ holds for all $n \geq 1$.

Since $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, Sandwich thm applies here and tells us that $\lim_{n\to\infty} \frac{\sin n}{n}$ exists and equals zero. $\square$

<u>Example 4.</u> For any $r > 0$, show that

$$\lim_{n\to\infty} r^{1/n} = 1.$$

<u>Solution</u>: Define $x_n = r^{1/n} - 1$. First, let us assume that $r \geq 1$. Then, $r^{1/n} \geq 1 \Rightarrow x_n \geq 0$ for all $n \geq 1$. Next, observe that

$$r = (1 + x_n)^n \geq 1 + n x_n$$

[We used the inequality $(1+x)^n \geq 1 + nx$, for $x \geq 0$. Why does this hold? Because

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n$$
$$\geq 1 + \binom{n}{1} x \quad (\because \text{other terms are} \geq 0).]$$

Therefore, we have

$$0 \leq x_n \leq \frac{r-1}{n} \quad \text{for all } n \geq 1.$$

Since $\lim_{n\to\infty} \frac{r-1}{n} = 0$, Sandwich applies and tells us that $\lim_{n\to\infty} x_n$ exists ~~ea~~ and equals zero.

$$\therefore \lim_{n\to\infty} (r^{1/n} - 1) = 0 \Rightarrow \lim_{n\to\infty} r^{1/n} = 1.$$

Next, let us consider the case $0 < r < 1$.

In this case, $\frac{1}{r} > 1$. Call $s = \frac{1}{r}$. Then

it follows that $\lim_{n\to\infty} s^{1/n} = 1$ ($\because s > 1$).

Hence, $\lim_{n\to\infty} \frac{1}{r^{1/n}} = 1 \Rightarrow \lim_{n\to\infty} r^{1/n} = 1$. $\square$

Example 5. Show that, $\lim_{n\to\infty} n^{1/n} = 1$.

Solution: Define $x_n = n^{1/n} - 1$. Since $n \geq 1$,

it follows that $x_n \geq 0$. Next, observe that

$$n = (1 + x_n)^n = 1 + \binom{n}{1} x_n + \binom{n}{2} x_n^2 + \dots + x_n^n$$
$$\geq 1 + \binom{n}{2} x_n^2 \quad (\because x_n \geq 0)$$

$$\Rightarrow x_n^2 \leq \frac{2}{n}$$

$$\therefore 0 \leq x_n \leq \sqrt{\frac{2}{n}} \text{ for all } n \geq 1.$$

Since $\lim_{n\to\infty} \sqrt{\frac{2}{n}} = 0$, Sandwich tells us that

$\lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} n^{1/n} = 1$. $\square$

Example 6. Find the limit $\lim_{n\to\infty} (2^n + 5^n)^{1/n}$.

Solution [Main idea: $(2^n + 5^n)^{1/n} = 5\left(1 + \underbrace{(2/5)^n}_{\text{small}}\right)^{1/n}$.]

Observe that $5^n < 2^n + 5^n < 2 \cdot 5^n$

$$\Rightarrow 5 < (2^n + 5^n)^{1/n} < 5 . 2^{1/n}.$$

Since $\lim_{n\to\infty} 5 \cdot 2^{1/n} = 5$ (thanks to example 4.),

Sandwich applies here and gives

$$\lim_{n\to\infty} (2^n + 5^n)^{1/n} = 5.$$

Example 7. Define $g : \mathbb{N} \to \mathbb{N}$ as: $g(n)$ = product of digits of n, written in base 10. Find the limit

$$\lim_{n\to\infty} \frac{g(n)}{n^{3/2}}.$$

Solution. Suppose n has k digits, with m being the first digit from left. Then,

$$g(n) \leq m \times 9^{k-1} < m \times 10^{k-1} \leq n.$$

Hence, $0 \leq \frac{g(n)}{n^{3/2}} \leq \frac{1}{\sqrt{n}}$ holds for all $n \geq 1$.

Sandwich gives $\lim_{n\to\infty} \frac{g(n)}{n^{3/2}} = 0$.

Example 8. Show that,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist otherwise} \end{cases}$$

Solution: If $r = 1$, then trivially $\lim_{n\to\infty} r^n = 1$. And if $r = -1$, then $\lim_{n\to\infty} r^n = \lim_{n\to\infty} (-1)^n$, does not exist.

Next, we consider two cases: $|r|<1$ or $|r|>1$.

$\boxed{\text{Case 1: } |r|<1}$ Need to show that $\lim_{n\to\infty} r^n = 0$.

We know, if $|x_n|$ converges to 0, then x_n must do the same. Hence it suffices to show $\lim_{n\to\infty} |r|^n = 0$.

If $r=0$, its obvious. So let $r \neq 0$ now. Then, we have $\frac{1}{|r|} > 1$. Call $\frac{1}{|r|} = s+1 \Rightarrow |r| = \frac{1}{s+1}$.

Now, since $s>0$, $(1+s)^n \geq 1+ns$; which gives

$$0 < \frac{1}{(1+s)^n} \leq \frac{1}{1+ns} < \frac{1}{ns}.$$

Sandwich applies here and tells us that

$$\lim_{n\to\infty} \frac{1}{(1+s)^n} = 0 \Rightarrow \lim_{n\to\infty} |r|^n = 0.$$

$\boxed{\text{Case 2: } |r|>1}$ Need to show that $\lim_{n\to\infty} r^n$ does not exist.

~~Let, if possible~~ Let us assume to the contrary that $\lim_{n\to\infty} r^n$ exists. Then $\lim_{n\to\infty} |r|^n$ exists, which implies that $\{|r|^n\}_{n\geq 1}$ is bounded. But, we have $|r|>1$, so $\beta = |r|-1 > 0$ and note that $|r|^n = (1+\beta)^n \geq 1+n\beta$, for all $n\geq 1$. Since $\beta > 0$ is fixed, $\{1+n\beta\}_{n\geq 1}$ is unbounded, hence $\{|r|^n\}_{n\geq 1}$ must be unbounded as well.

~~So we reached a~~ So we get a contradiction.

Problems

1. Prove that $\lim_{n\to\infty} \{\sqrt{n^2+n+1}\}$ exists, where $\{x\}$ is defined as: $\{x\} = x - [x]$ = fractional part of x.
Also, find the above limit.

2. For $x > 0$, show that the following limit exists and also evaluate it —
$$\lim_{n\to\infty} \frac{[x]+[2x]+\dots+[nx]}{n^2}.$$
(Here, $[x]$ = largest integer $\le x$, as in problem 1.)

3. Let $f(n)$ denote no. of digits of 8^n, when it is written in base 6. And let $g(n)$ denote no. of digits of 6^n when it is written in base 8.
Show that $\lim_{n\to\infty} \frac{f(n)\,g(n)}{n^2}$ exists and also find this limit.
Hint: If m has k digits in base b, then
$b^{k-1} \le m \le (b-1)(b^{k-1} + b^{k-2} + \dots + b+1) = b^k - 1$,
which gives $k-1 \le \log_b m < k$. $\therefore$ $k = [\log_b m] + 1$.

4. Suppose $0 \le a_1 \le a_2 \le \dots \le a_r$ are fixed real numbers. Show that the following limit exists and also evaluate it:
$$\lim_{n\to\infty} (a_1^n + a_2^n + \dots + a_r^n)^{\frac{1}{n}}.$$

5. Show that $\lim_{n\to\infty} \sqrt[n]{1^2+2^2+\dots+n^2}$ exists, and also find this limit.

6. Evaluate the limit (after showing it exists)

$$\lim_{n\to\infty} \left(\frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \dots + \frac{n}{n^2+n+n}\right).$$

7. Suppose $\{a_n\}_{n\geq 1}$ satisfies $\frac{2n^2-7}{4n+5} < a_n < \frac{3n^2+8}{6n-1}$ for all $n \geq 1$. Find $\lim_{n\to\infty} \frac{n\,a_n}{(n+1)^2}$.

8. (a) Suppose $x_n \to x$ as $n\to\infty$. Show that $\sin x_n \to \sin x$ and $\cos x_n \to \cos x$, as $n\to\infty$. (You can use the following fact: for every $x \geq 0$, $\sin x \leq x$ holds.)

(b) Using $\sin x \leq x \leq \tan x$ for $x \in (0, \frac{\pi}{2})$, show that $\lim_{n\to\infty} n \sin\frac{1}{n} = 1$.

(c) Suppose $\{a_n\}_{n\geq 1}$ is a sequence that satisfies

$$\sin\frac{1}{n+1} < a_n < \sin\frac{1}{n} \quad \text{for all } n \geq 1.$$

Show that $\lim_{n\to\infty} n\,a_n = 1$.

9.* Show that $\lim_{n\to\infty} \frac{1}{\log(1+\frac{1}{n})}\left(\sum_{k=n}^{\infty} \frac{1}{k^2}\right) = 1$.

[Hint: Use $\frac{1}{(k+1)k} \leq \frac{1}{k^2} \leq \frac{1}{(k-1)k}$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$.]

(* = it uses tools not covered yet.)