Chapter 4

Series

> Divergent series are the devil, and it is a shame to base on them any demonstration whatsoever. (Niels Henrik Abel, 1826)
>
> This series is divergent, therefore we may be able to do something with it. (Oliver Heaviside, quoted by Kline)

In this chapter, we apply our results for sequences to series, or infinite sums. The convergence and sum of an infinite series is defined in terms of its sequence of finite partial sums.

4.1. Convergence of series

A finite sum of real numbers is well-defined by the algebraic properties of $\mathbb{R}$, but in order to make sense of an infinite series, we need to consider its convergence. We say that a series converges if its sequence of partial sums converges, and in that case we define the sum of the series to be the limit of its partial sums.

Definition 4.1. Let (a_n) be a sequence of real numbers. The series

$$\sum_{n=1}^{\infty} a_n$$

converges to a sum $S \in \mathbb{R}$ if the sequence (S_n) of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

converges to S as $n \to \infty$. Otherwise, the series diverges.

If a series converges to S, we write

$$S = \sum_{n=1}^{\infty} a_n.$$

We also say a series diverges to $\pm\infty$ if its sequence of partial sums does. As for sequences, we may start a series at other values of n than $n = 1$ without changing its convergence properties. It is sometimes convenient to omit the limits on a series when they aren't important, and write it as $\sum a_n$.

Example 4.2. If $|a| < 1$, then the geometric series with ratio a converges and its sum is

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

This series is simple enough that we can compute its partial sums explicitly,

$$S_n = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

As shown in Proposition 3.31, if $|a| < 1$, then $a^n \to 0$ as $n \to \infty$, so that $S_n \to 1/(1-a)$, which proves the result.

The geometric series diverges to ∞ if $a \geq 1$, and diverges in an oscillatory fashion if $a \leq -1$. The following examples consider the cases $a = \pm 1$ in more detail.

Example 4.3. The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \dots$$

diverges to ∞, since its nth partial sum is $S_n = n$.

Example 4.4. The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \dots$$

diverges, since its partial sums

$$S_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

oscillate between 0 and 1.

This series illustrates the dangers of blindly applying algebraic rules for finite sums to series. For example, one might argue that

$$S = (1-1) + (1-1) + (1-1) + \dots = 0 + 0 + 0 + \dots = 0,$$

or that

$$S = 1 + (-1+1) + (-1+1) + \dots = 1 + 0 + 0 + \dots = 1,$$

or that

$$1 - S = 1 - (1 - 1 + 1 - 1 + \dots) = 1 - 1 + 1 - 1 + 1 - \dots = S,$$

so $2S = 1$ or $S = 1/2$.

The Italian mathematician and priest Luigi Grandi (1710) suggested that these results were evidence in favor of the existence of God, since they showed that it was possible to create something out of nothing.

Telescoping series of the form

$$\sum_{n=1}^{\infty}(a_n - a_{n+1})$$

are another class of series whose partial sums

$$S_n = a_1 - a_{n+1}$$

can be computed explicitly and then used to study their convergence. We give one example.

Example 4.5. The series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots$$

converges to 1. To show this, we observe that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k(k+1)} &= \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)\\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1}\\ &= 1 - \frac{1}{n+1},\end{aligned}$$

and it follows that

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1.$$

A condition for the convergence of series with positive terms follows immediately from the condition for the convergence of monotone sequences.

Proposition 4.6. A series $\sum a_n$ with positive terms $a_n \geq 0$ converges if and only if its partial sums

$$\sum_{k=1}^{n} a_k \leq M$$

are bounded from above, otherwise it diverges to ∞.

Proof. The partial sums $S_n = \sum_{k=1}^{n} a_k$ of such a series form a monotone increasing sequence, and the result follows immediately from Theorem 3.29 □

Although we have only defined sums of convergent series, divergent series are not necessarily meaningless. For example, the Cesàro sum C of a series $\sum a_n$ is defined by

$$C = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} S_n, \qquad S_n = a_1 + a_2 + \dots + a_n.$$

That is, we average the first n partial sums the series, and let $n \to \infty$. One can prove that if a series converges to S, then its Cesàro sum exists and is equal to S, but a series may be Cesàro summable even if it is divergent.

Example 4.7. For the series $\sum(-1)^{n+1}$ in Example 4.4, we find that

$$\frac{1}{n}\sum_{k=1}^{n} S_k = \begin{cases} 1/2 + 1/(2n) & \text{if } n \text{ is odd,} \\ 1/2 & \text{if } n \text{ is even,} \end{cases}$$

since the S_n's alternate between 0 and 1. It follows the Cesàro sum of the series is $C = 1/2$. This is, in fact, what Grandi believed to be the "true" sum of the series.

Cesàro summation is important in the theory of Fourier series. There are also many other ways to sum a divergent series or assign a meaning to it (for example, as an asymptotic series), but we won't discuss them further here.

4.2. The Cauchy condition

The following Cauchy condition for the convergence of series is an immediate consequence of the Cauchy condition for the sequence of partial sums.

Theorem 4.8 (Cauchy condition)**.** The series

$$\sum_{n=1}^{\infty} a_n$$

converges if and only for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left|\sum_{k=m+1}^{n} a_k\right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon \qquad \text{for all } n > m > N.$$

Proof. The series converges if and only if the sequence (S_n) of partial sums is Cauchy, meaning that for every $\epsilon > 0$ there exists N such that

$$|S_n - S_m| = \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon \qquad \text{for all } n > m > N,$$

which proves the result. □

A special case of this theorem is a necessary condition for the convergence of a series, namely that its terms approach zero. This condition is the first thing to check when considering whether or not a given series converges.

Theorem 4.9. If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. If the series converges, then it is Cauchy. Taking $m = n - 1$ in the Cauchy condition in Theorem 4.8, we find that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n > N$, which proves that $a_n \to 0$ as $n \to \infty$. □

Example 4.10. The geometric series $\sum a^n$ converges if $|a| < 1$ and in that case $a^n \to 0$ as $n \to \infty$. If $|a| \geq 1$, then $a^n \not\to 0$ as $n \to \infty$, which implies that the series diverges.

The condition that the terms of a series approach zero is not, however, sufficient to imply convergence. The following series is a fundamental example.

Example 4.11. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges, even though $1/n \to 0$ as $n \to \infty$. To see this, we collect the terms in successive groups of powers of two,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) + \dots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16}\right) + \dots \\ &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots .\end{aligned}$$

In general, for every $n \geq 1$, we have

$$\begin{aligned}\sum_{k=1}^{2^{n+1}} \frac{1}{k} &= 1 + \frac{1}{2} + \sum_{j=1}^{n} \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\ &> 1 + \frac{1}{2} + \sum_{j=1}^{n} \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{2^{j+1}} \\ &> 1 + \frac{1}{2} + \sum_{j=1}^{n} \frac{1}{2} \\ &> \frac{n}{2} + \frac{3}{2},\end{aligned}$$

so the series diverges. We can similarly obtain an upper bound for the partial sums,

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} < 1 + \frac{1}{2} + \sum_{j=1}^{n} \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{2^j} < n + \frac{3}{2}.$$

These inequalities are rather crude, but they show that the series diverges at a logarithmic rate, since the sum of 2^n terms is of the order n. This rate of divergence is very slow. It takes 12367 terms for the partial sums of harmonic series to exceed 10, and more than 1.5×10^{43} terms for the partial sums to exceed 100.

A more refined argument, using integration, shows that

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n\right] = \gamma$$

where $\gamma \approx 0.5772$ is the Euler constant. (See Example 12.45.)

4.3. Absolutely convergent series

There is an important distinction between absolutely and conditionally convergent series.

Definition 4.12. The series

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if

$$\sum_{n=1}^{\infty} |a_n| \text{ converges,}$$

and converges conditionally if

$$\sum_{n=1}^{\infty} a_n \text{ converges, but } \sum_{n=1}^{\infty} |a_n| \text{ diverges.}$$

We will show in Proposition 4.17 below that every absolutely convergent series converges. For series with positive terms, there is no difference between convergence and absolute convergence. Also note from Proposition 4.6 that $\sum a_n$ converges absolutely if and only if the partial sums $\sum_{k=1}^{n} |a_k|$ are bounded from above.

Example 4.13. The geometric series $\sum a^n$ is absolutely convergent if $|a| < 1$.

Example 4.14. The alternating harmonic series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is not absolutely convergent since, as shown in Example 4.11, the harmonic series diverges. It follows from Theorem 4.30 below that the alternating harmonic series converges, so it is a conditionally convergent series. Its convergence is made possible by the cancelation between terms of opposite signs.

As we show next, the convergence of an absolutely convergent series follows from the Cauchy condition. Moreover, the series of positive and negative terms in an absolutely convergent series converge separately. First, we introduce some convenient notation.

Definition 4.15. The positive and negative parts of a real number $a \in \mathbb{R}$ are given by

$$a^+ = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a \le 0, \end{cases} \qquad a^- = \begin{cases} 0 & \text{if } a \ge 0, \\ |a| & \text{if } a < 0. \end{cases}$$

It follows, in particular, that

$$0 \le a^+, a^- \le |a|, \qquad a = a^+ - a^-, \qquad |a| = a^+ + a^-.$$

We may then split a series of real numbers into its positive and negative parts.

Example 4.16. Consider the alternating harmonic series

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

Its positive and negative parts are given by

$$\sum_{n=1}^{\infty} a_n^+ = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + 0 + \dots,$$
$$\sum_{n=1}^{\infty} a_n^- = 0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \frac{1}{6} + \dots.$$

Both of these series diverge to infinity, since the harmonic series diverges and

$$\sum_{n=1}^{\infty} a_n^+ > \sum_{n=1}^{\infty} a_n^- = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}.$$

Proposition 4.17. An absolutely convergent series converges. Moreover,

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if and only if the series

$$\sum_{n=1}^{\infty} a_n^+, \qquad \sum_{n=1}^{\infty} a_n^-$$

of positive and negative terms both converge. Furthermore, in that case

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-, \qquad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^-.$$

Proof. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, so it satisfies the Cauchy condition. Since

$$\left|\sum_{k=m+1}^{n} a_k\right| \le \sum_{k=m+1}^{n} |a_k|,$$

the series $\sum a_n$ also satisfies the Cauchy condition, and therefore it converges.

For the second part, note that

$$0 \le \sum_{k=m+1}^{n} |a_k| = \sum_{k=m+1}^{n} a_k^+ + \sum_{k=m+1}^{n} a_k^-,$$
$$0 \le \sum_{k=m+1}^{n} a_k^+ \le \sum_{k=m+1}^{n} |a_k|,$$
$$0 \le \sum_{k=m+1}^{n} a_k^- \le \sum_{k=m+1}^{n} |a_k|,$$

which shows that $\sum |a_n|$ is Cauchy if and only if both $\sum a_n^+$, $\sum a_n^-$ are Cauchy. It follows that $\sum |a_n|$ converges if and only if both $\sum a_n^+$, $\sum a_n^-$ converge. In that case, we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^- \right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k^+ - \lim_{n\to\infty} \sum_{k=1}^{n} a_k^- \\ &= \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-,\end{aligned}$$

and similarly for $\sum |a_n|$, which proves the proposition. □

It is worth noting that this result depends crucially on the completeness of $\mathbb{R}$.

Example 4.18. Suppose that $a_n^+, a_n^- \in \mathbb{Q}^+$ are positive rational numbers such that

$$\sum_{n=1}^{\infty} a_n^+ = \sqrt{2}, \qquad \sum_{n=1}^{\infty} a_n^- = 2 - \sqrt{2},$$

and let $a_n = a_n^+ - a_n^-$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- = 2\sqrt{2} - 2 \notin \mathbb{Q}, \\ \sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^- = 2 \in \mathbb{Q}.\end{aligned}$$

Thus, the series converges absolutely in $\mathbb{Q}$, but it doesn't converge in $\mathbb{Q}$.

4.4. The comparison test

One of the most useful ways of showing that a series is absolutely convergent is to compare it with a simpler series whose convergence is already known.

Theorem 4.19 (Comparison test)**.** Suppose that $b_n \geq 0$ and

$$\sum_{n=1}^{\infty} b_n$$

converges. If $|a_n| \leq b_n$, then

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely.

Proof. Since $\sum b_n$ converges it satisfies the Cauchy condition, and since

$$\sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} b_k$$

the series $\sum |a_n|$ also satisfies the Cauchy condition. Therefore $\sum a_n$ converges absolutely. □

Example 4.20. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots.$$

converges by comparison with the telescoping series in Example 4.5. We have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

and

$$0 \le \frac{1}{(n+1)^2} < \frac{1}{n(n+1)}.$$

We also get the explicit upper bound

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2.$$

In fact, the sum is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Mengoli (1644) posed the problem of finding this sum, which was solved by Euler (1735). The evaluation of the sum is known as the Basel problem, perhaps after Euler's birthplace in Switzerland.

Example 4.21. The series in Example 4.20 is a special case of the following series, called the p-series,

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where $0 < p < \infty$. It follows by comparison with the harmonic series in Example 4.11 that the p-series diverges for $p \le 1$. (If it converged for some $p \le 1$, then the harmonic series would also converge.) On the other hand, the p-series converges for every $1 < p < \infty$. To show this, note that

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{2}{2^p}, \qquad \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{4}{4^p},$$

and so on, which implies that

$$\sum_{n=1}^{2^N-1} \frac{1}{n^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \dots + \frac{1}{2^{(N-1)(p-1)}} < \frac{1}{1-2^{1-p}}.$$

Thus, the partial sums are bounded from above, so the series converges by Proposition 4.6. An alternative proof of the convergence of the p-series for $p > 1$ and divergence for $0 < p \le 1$, using the integral test, is given in Example 12.44.