4.5. * The Riemann ζ-function

Example 4.21 justifies the following definition.

Definition 4.22. The Riemann ζ-function is defined for $1 < s < \infty$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

For instance, as stated in Example 4.20, we have $\zeta(2) = \pi^2/6$. In fact, Euler (1755) discovered a general formula for the value $\zeta(2n)$ of the ζ-function at even natural numbers,

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \qquad n = 1, 2, 3, \dots,$$

where the coefficients B_{2n} are the Bernoulli numbers (see Example 10.19). In particular,

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555}.$$

On the other hand, the values of the ζ-function at odd natural numbers are harder to study. For instance,

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.2020569\dots$$

is called Apéry's constant. It was proved to be irrational by Apéry (1979) but a simple explicit expression for $\zeta(3)$ is not known (and likely doesn't exist).

The Riemann ζ-function is intimately connected with number theory and the distribution of primes. Every positive integer n has a unique factorization

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k},$$

where the p_j are primes and the exponents α_j are positive integers. Using the binomial expansion in Example 4.2, we have

$$\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \dots.$$

By expanding the products and rearranging the resulting sums, one can see that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is taken over all primes p, since every possible prime factorization of a positive integer appears exactly once in the sum on the right-hand side. The infinite product here is defined as a limit of finite products,

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \lim_{N\to\infty} \prod_{p \le N} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Using complex analysis, one can show that the ζ-function may be extended in a unique way to an analytic (i.e., differentiable) function of a complex variable $s = \sigma + it \in \mathbb{C}$

$$\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C},$$

where $\sigma = \Re s$ is the real part of s and $t = \Im s$ is the imaginary part. The ζ-function has a singularity at $s = 1$, called a simple pole, where it goes to infinity like $1/(1-s)$, and is equal to zero at the negative even integers $s = -2, -4, \dots, -2n, \dots$. These zeros are called the trivial zeros of the ζ-function. Riemann (1859) made the following conjecture.

Hypothesis 4.23 (Riemann hypothesis)**.** Except for the trivial zeros, the only zeros of the Riemann ζ-function occur on the line $\Re s = 1/2$.

If true, the Riemann hypothesis has significant consequences for the distribution of primes (and many other things); roughly speaking, it implies that the prime numbers are "randomly distributed" among the natural numbers (with density $1/\log n$ near a large integer $n \in \mathbb{N}$). Despite enormous efforts, this conjecture has neither been proved nor disproved, and it remains one of the most significant open problems in mathematics (perhaps *the* most significant open problem).

4.6. The ratio and root tests

In this section, we describe the ratio and root tests, which provide explicit sufficient conditions for the absolute convergence of a series that can be compared with a geometric series. These tests are particularly useful in studying power series, but they aren't effective in determining the convergence or divergence of series whose terms do not approach zero at a geometric rate.

Theorem 4.24 (Ratio test)**.** Suppose that (a_n) is a sequence of nonzero real numbers such that the limit

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists or diverges to infinity. Then the series

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if $0 \le r < 1$ and diverges if $1 < r \le \infty$.

Proof. If $r < 1$, choose s such that $r < s < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| < s \qquad \text{for all } n > N.$$

It follows that

$$|a_n| \le M s^n \qquad \text{for all } n > N$$

where M is a suitable constant. Therefore $\sum a_n$ converges absolutely by comparison with the convergent geometric series $\sum M s^n$.

If $r > 1$, choose s such that $r > s > 1$. There exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| > s \qquad \text{for all } n > N,$$

so that $|a_n| \geq Ms^n$ for all $n > N$ and some $M > 0$. It follows that (a_n) does not approach 0 as $n \to \infty$, so the series diverges. □

Example 4.25. Let $a \in \mathbb{R}$, and consider the series

$$\sum_{n=1}^{\infty} na^n = a + 2a^2 + 3a^3 + \dots.$$

Then

$$\lim_{n\to\infty} \left|\frac{(n+1)a^{n+1}}{na^n}\right| = |a| \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = |a|.$$

By the ratio test, the series converges if $|a| < 1$ and diverges if $|a| > 1$; the series also diverges if $|a| = 1$. The convergence of the series for $|a| < 1$ is explained by the fact that the geometric decay of the factor a^n is more rapid than the algebraic growth of the coefficient n.

Example 4.26. Let $p > 0$ and consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Then

$$\lim_{n\to\infty} \left[\frac{1/(n+1)^p}{1/n^p}\right] = \lim_{n\to\infty} \left[\frac{1}{(1+1/n)^p}\right] = 1,$$

so the ratio test is inconclusive. In this case, the series diverges if $0 < p \leq 1$ and converges if $p > 1$, which shows that either possibility may occur when the limit in the ratio test is 1.

The root test provides a criterion for convergence of a series that is closely related to the ratio test, but it doesn't require that the limit of the ratios of successive terms exists.

Theorem 4.27 (Root test)**.** Suppose that (a_n) is a sequence of real numbers and let

$$r = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then the series

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if $0 \leq r < 1$ and diverges if $1 < r \leq \infty$.

Proof. First suppose $0 \leq r < 1$. If $0 < r < 1$, choose s such that $r < s < 1$, and let

$$t = \frac{r}{s}, \qquad r < t < 1.$$

If $r = 0$, choose any $0 < t < 1$. Since $t > \limsup |a_n|^{1/n}$, Theorem 3.41 implies that there exists $N \in \mathbb{N}$ such that

$$|a_n|^{1/n} < t \qquad \text{for all } n > N.$$

Therefore $|a_n| < t^n$ for all $n > N$, where $t < 1$, so it follows that the series converges by comparison with the convergent geometric series $\sum t^n$.

Next suppose $1 < r \leq \infty$. If $1 < r < \infty$, choose s such that $1 < s < r$, and let

$$t = \frac{r}{s}, \qquad 1 < t < r.$$

If $r = \infty$, choose any $1 < t < \infty$. Since $t < \limsup |a_n|^{1/n}$, Theorem 3.41 implies that

$$|a_n|^{1/n} > t \qquad \text{for infinitely many } n \in \mathbb{N}.$$

Therefore $|a_n| > t^n$ for infinitely many $n \in \mathbb{N}$, where $t > 1$, so (a_n) does not approach zero as $n \to \infty$, and the series diverges. □

The root test may succeed where the ratio test fails.

Example 4.28. Consider the geometric series with ratio 1/2,

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \dots, \qquad a_n = \frac{1}{2^n}.$$

Then (of course) both the ratio and root test imply convergence since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{2} < 1.$$

Now consider the series obtained by switching successive odd and even terms

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2^2} + \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^6} + \dots, \qquad b_n = \begin{cases} 1/2^{n+1} & \text{if } n \text{ is odd,} \\ 1/2^{n-1} & \text{if } n \text{ is even} \end{cases}$$

For this series,

$$\left| \frac{b_{n+1}}{b_n} \right| = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1/8 & \text{if } n \text{ is even,} \end{cases}$$

and the ratio test doesn't apply, since the required limit does not exist. (The series still converges at a geometric rate, however, because the the decrease in the terms by a factor of 1/8 for even n dominates the increase by a factor of 2 for odd n.) On the other hand

$$\limsup_{n\to\infty} |b_n|^{1/n} = \frac{1}{2},$$

so the ratio test still works. In fact, as we discuss in Section 4.8, since the series is absolutely convergent, every rearrangement of it converges to the same sum.

4.7. Alternating series

An alternating series is one in which successive terms have opposite signs. If the terms in an alternating series have decreasing absolute values and converge to zero, then the series converges however slowly its terms approach zero. This allows us to prove the convergence of some series which aren't absolutely convergent.

Example 4.29. The alternating harmonic series from Example 4.14 is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots.$$

The behavior of its partial sums is shown in Figure 1, which illustrates the idea of the convergence proof for alternating series.

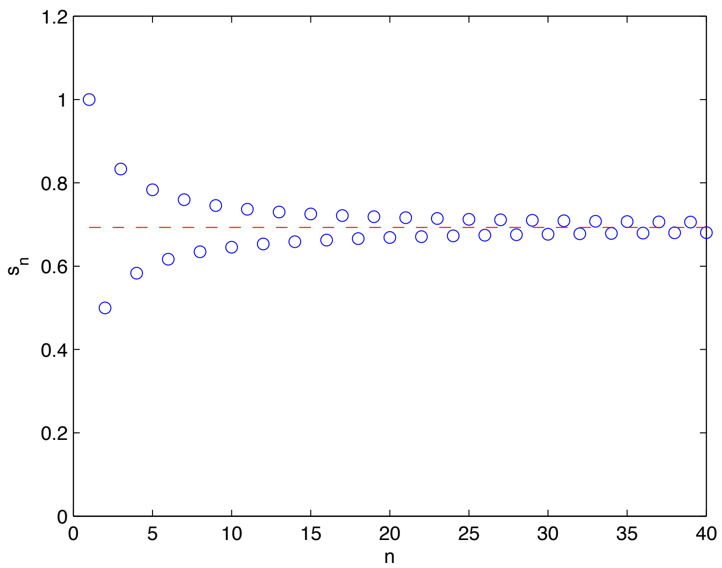

Figure 1. A plot of the first 40 partial sums S_n of the alternating harmonic series in Example 4.14. The odd partial sums decrease and the even partial sums increase to the sum of the series $\log 2 \approx 0.6931$, which is indicated by the dashed line.

Theorem 4.30 (Alternating series)**.** Suppose that (a_n) is a decreasing sequence of nonnegative real numbers, meaning that $0 \le a_{n+1} \le a_n$, such that $a_n \to 0$ as $n \to \infty$. Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

converges.

Proof. Let

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$

denote the nth partial sum. If $n = 2m-1$ is odd, then

$$S_{2m-1} = S_{2m-3} - a_{2m-2} + a_{2m-1} \le S_{2m-3},$$

since (a_n) is decreasing, and

$$S_{2m-1} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2m-3} - a_{2m-2}) + a_{2m-1} \ge 0.$$

Thus, the sequence (S_{2m-1}) of odd partial sums is decreasing and bounded from below by 0, so $S_{2m-1} \downarrow S^+$ as $m \to \infty$ for some $S^+ \ge 0$.

Similarly, if $n = 2m$ is even, then

$$S_{2m} = S_{2m-2} + a_{2m-1} - a_{2m} \ge S_{2m-2},$$

and

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2m-1} - a_{2m}) \le a_1.$$

Thus, (S_{2m}) is increasing and bounded from above by a_1, so $S_{2m} \uparrow S^- \le a_1$ as $m \to \infty$.

Finally, note that

$$\lim_{m\to\infty} (S_{2m-1} - S_{2m}) = \lim_{m\to\infty} a_{2m} = 0,$$

so $S^+ = S^-$, which implies that the series converges to their common value. □

The proof also shows that the sum $S_{2m} \le S \le S_{2n-1}$ is bounded from below and above by all even and odd partial sums, respectively, and that the error $|S_n - S|$ is less than the first term a_{n+1} in the series that is neglected.

Example 4.31. The alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

converges for every $p > 0$. The convergence is absolute for $p > 1$ and conditional for $0 < p \le 1$.

4.8. Rearrangements

A rearrangement of a series is a series that consists of the same terms in a different order. The convergence of rearranged series may initially appear to be unconnected with absolute convergence, but absolutely convergent series are exactly those series whose sums remain the same under every rearrangement of their terms. On the other hand, a conditionally convergent series can be rearranged to give any sum we please, or to diverge.

Example 4.32. A rearrangement of the alternating harmonic series in Example 4.14 is

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots,$$

where we put two negative even terms between each of the positive odd terms. The behavior of its partial sums is shown in Figure 2. As proved in Example 12.47, this series converges to one-half of the sum of the alternating harmonic series. The sum of the alternating harmonic series can change under rearrangement because it is conditionally convergent.

Note also that both the positive and negative parts of the alternating harmonic series diverge to infinity, since

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots &> \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots \\ &> \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right), \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right), \end{aligned}$$

and the harmonic series diverges. This is what allows us to change the sum by rearranging the series.

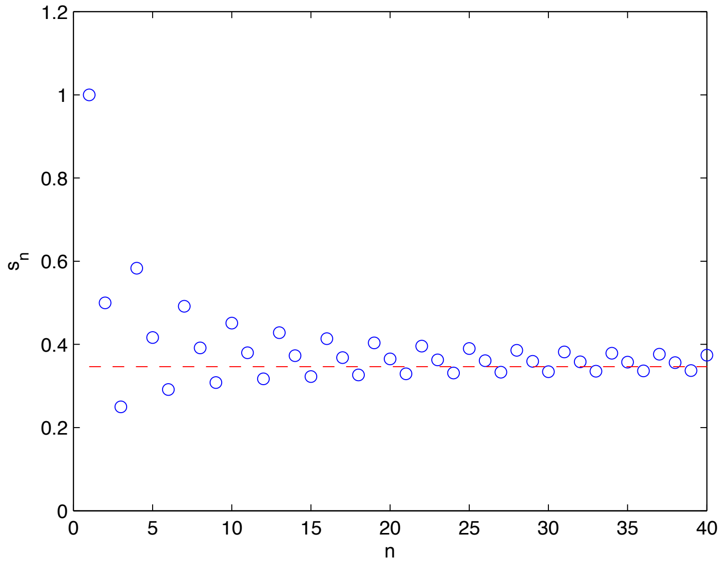

Figure 2. A plot of the first 40 partial sums S_n of the rearranged alternating harmonic series in Example 4.32. The series converges to half the sum of the alternating harmonic series, $\frac{1}{2}\log 2 \approx 0.3466$. Compare this picture with Figure 1.

The formal definition of a rearrangement is as follows.

Definition 4.33. A series

$$\sum_{m=1}^{\infty} b_m$$

is a rearrangement of a series

$$\sum_{n=1}^{\infty} a_n$$

if there is a one-to-one, onto function $f : \mathbb{N} \to \mathbb{N}$ such that $b_m = a_{f(m)}$.

If $\sum b_m$ is a rearrangement of $\sum a_n$ with $n = f(m)$, then $\sum a_n$ is a rearrangement of $\sum b_m$, with $m = f^{-1}(n)$.

Theorem 4.34. If a series is absolutely convergent, then every rearrangement of the series converges to the same sum.

Proof. First, suppose that

$$\sum_{n=1}^{\infty} a_n$$

is a convergent series with $a_n \geq 0$, and let

$$\sum_{m=1}^{\infty} b_m, \qquad b_m = a_{f(m)}$$

be a rearrangement.

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

$$0 \leq \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{N} a_k < \epsilon.$$

Since $f : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto, there exists $M \in \mathbb{N}$ such that

$$\{1, 2, \ldots, N\} \subset f^{-1}\left(\{1, 2, \ldots, M\}\right),$$

meaning that all of the terms a_1, a_2,..., a_N are included among the b_1, b_2,..., b_M. For example, we can take $M = \max\{m \in \mathbb{N} : 1 \leq f(m) \leq N\}$; this maximum is well-defined since there are finitely many such m (in fact, N of them).

If $m > M$, then

$$\sum_{k=1}^{N} a_k \leq \sum_{j=1}^{m} b_j \leq \sum_{k=1}^{\infty} a_k$$

since the b_j's include all the a_k's in the left sum, all the b_j's are included among the a_k's in the right sum, and $a_k, b_j \geq 0$. It follows that

$$0 \leq \sum_{k=1}^{\infty} a_k - \sum_{j=1}^{m} b_j < \epsilon,$$

for all $m > M$, which proves that

$$\sum_{j=1}^{\infty} b_j = \sum_{k=1}^{\infty} a_k.$$

If $\sum a_n$ is a general absolutely convergent series, then from Proposition 4.17 the positive and negative parts of the series

$$\sum_{n=1}^{\infty} a_n^+, \qquad \sum_{n=1}^{\infty} a_n^-$$

converge. If $\sum b_m$ is a rearrangement of $\sum a_n$, then $\sum b_m^+$ and $\sum b_m^-$ are rearrangements of $\sum a_n^+$ and $\sum a_n^-$, respectively. It follows from what we've just proved that they converge and

$$\sum_{m=1}^{\infty} b_m^+ = \sum_{n=1}^{\infty} a_n^+, \qquad \sum_{m=1}^{\infty} b_m^- = \sum_{n=1}^{\infty} a_n^-.$$

Proposition 4.17 then implies that $\sum b_m$ is absolutely convergent and

$$\sum_{m=1}^{\infty} b_m = \sum_{m=1}^{\infty} b_m^+ - \sum_{m=1}^{\infty} b_m^- = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- = \sum_{n=1}^{\infty} a_n,$$

which proves the result. □

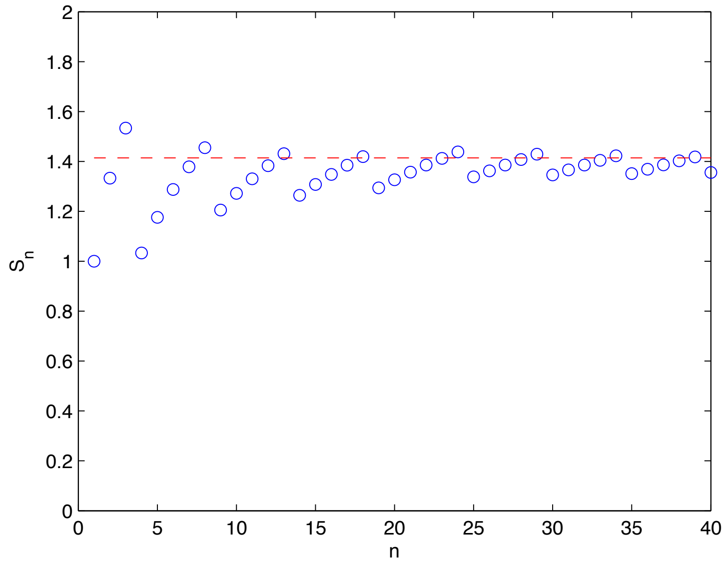

Figure 3. A plot of the first 40 partial sums S_n of the rearranged alternating harmonic series described in Example 4.35, which converges to $\sqrt{2}$.

Conditionally convergent series behave completely differently from absolutely convergent series under rearrangement. As Riemann observed, they can be rearranged to give any sum we want, or to diverge. Before giving the proof, we illustrate the idea with an example.

Example 4.35. Suppose we want to rearrange the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots.$$

so that its sum is $\sqrt{2} \approx 1.4142$. We choose positive terms until we get a partial sum that is greater than $\sqrt{2}$, which gives $1+1/3+1/5$; followed by negative terms until we get a sum less than $\sqrt{2}$, which gives $1 + 1/3 + 1/5 - 1/2$; followed by positive terms until we get a sum greater than $\sqrt{2}$, which gives

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13};$$

followed by another negative term $-1/4$ to get a sum less than $\sqrt{2}$; and so on. The first 40 partial sums of the resulting series are shown in Figure 3.

Theorem 4.36. If a series is conditionally convergent, then it has rearrangements that converge to an arbitrary real number and rearrangements that diverge to ∞ or $-\infty$.

Proof. Suppose that $\sum a_n$ is conditionally convergent. Since the series converges, $a_n \to 0$ as $n \to \infty$. If both the positive part $\sum a_n^+$ and negative part $\sum a_n^-$ of the series converge, then the series converges absolutely; and if only one part diverges, then the series diverges (to ∞ if $\sum a_n^+$ diverges, or $-\infty$ if $\sum a_n^-$ diverges). Therefore both $\sum a_n^+$ and $\sum a_n^-$ diverge. This means that we can make sums of successive positive or negative terms in the series as large as we wish.

Suppose $S \in \mathbb{R}$. Starting from the beginning of the series, we choose successive positive or zero terms in the series until their partial sum is greater than or equal to S. Then we choose successive strictly negative terms, starting again from the beginning of the series, until the partial sum of all the terms is strictly less than S. After that, we choose successive positive or zero terms until the partial sum is greater than or equal S, followed by negative terms until the partial sum is strictly less than S, and so on. The partial sums are greater than S by at most the value of the last positive term retained, and are less than S by at most the value of the last negative term retained. Since $a_n \to 0$ as $n \to \infty$, it follows that the rearranged series converges to S.

A similar argument shows that we can rearrange a conditional convergent series to diverge to ∞ or $-\infty$, and that we can rearrange the series so that it diverges in a finite or infinite oscillatory fashion. □

The previous results indicate that conditionally convergent series behave in many ways more like divergent series than absolutely convergent series.

4.9. The Cauchy product

In this section, we prove a result about the product of absolutely convergent series that is useful in multiplying power series. It is convenient to begin numbering the terms of the series at $n = 0$.

Definition 4.37. The Cauchy product of the series

$$\sum_{n=0}^{\infty} a_n, \qquad \sum_{n=0}^{\infty} b_n$$

is the series

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$

The Cauchy product arises formally by term-by-term multiplication and rearrangement:

$$\begin{aligned}
(a_0 + a_1 + a_2 + a_3 + \dots)\,&(b_0 + b_1 + b_2 + b_3 + \dots) \\
&= a_0b_0 + a_0b_1 + a_0b_2 + a_0b_3 + \dots + a_1b_0 + a_1b_1 + a_1b_2 + \dots \\
&\qquad + a_2b_0 + a_2b_1 + \dots + a_3b_0 + \dots \\
&= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) \\
&\qquad + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0) + \dots .
\end{aligned}$$

In general, writing $m = n - k$, we have formally that

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k b_m = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}.$$

There are no convergence issues about the individual terms in the Cauchy product, since $\sum_{k=0}^{n} a_k b_{n-k}$ is a finite sum.

Theorem 4.38 (Cauchy product)**.** If the series

$$\sum_{n=0}^{\infty} a_n, \qquad \sum_{n=0}^{\infty} b_n$$

are absolutely convergent, then the Cauchy product is absolutely convergent and

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

Proof. For every $N \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{n=0}^{N} \left| \sum_{k=0}^{n} a_k b_{n-k} \right| &\leq \sum_{n=0}^{N} \left(\sum_{k=0}^{n} |a_k||b_{n-k}| \right) \\ &\leq \left(\sum_{k=0}^{N} |a_k| \right) \left(\sum_{m=0}^{N} |b_m| \right) \\ &\leq \left(\sum_{n=0}^{\infty} |a_n| \right) \left(\sum_{n=0}^{\infty} |b_n| \right). \end{aligned}$$

Thus, the Cauchy product is absolutely convergent, since the partial sums of its absolute values are bounded from above.

Since the series for the Cauchy product is absolutely convergent, any rearrangement of it converges to the same sum. In particular, the subsequence of partial sums given by

$$\left(\sum_{n=0}^{N} a_n \right) \left(\sum_{n=0}^{N} b_n \right) = \sum_{n=0}^{N} \sum_{m=0}^{N} a_n b_m$$

corresponds to a rearrangement of the Cauchy product, so

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \lim_{N \to \infty} \left(\sum_{n=0}^{N} a_n \right) \left(\sum_{n=0}^{N} b_n \right) = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

□

In fact, as we discuss in the next section, since the series of term-by-term products of absolutely convergent series converges absolutely, every rearrangement of the product series — not just the one in the Cauchy product — converges to the product of the sums.

4.10. * Double series

A double series is a series of the form

$$\sum_{m,n=1}^{\infty} a_{mn},$$