# Solutions to Class Test on Calculus

Aditya Ghosh

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1. *Define* $x_n = \sin a + \sin(a+d) + \sin(a+2d) + \cdots + \sin(a+(n-1)d)$, *for* $n \geq 1$. *Find all real numbers* $a, d$ *for which the sequence* $\{x_n\}_{n\geq 1}$ *is bounded.*

   Solution: First let us assume that $d/2 \neq k\pi$ for any $k \in \mathbb{Z}$, which ensures that $\sin\frac{d}{2} \neq 0$. We observe that

$$
\begin{aligned}
2\sin\frac{d}{2}x_n &= 2\sin\frac{d}{2}\sin a + 2\sin\frac{d}{2}\sin(a+d) + \cdots + 2\sin\frac{d}{2}\sin(a+(n-1)d) \\
&= \cos\left(a-\frac{d}{2}\right) - \cos\left(a+\frac{d}{2}\right) + \cos\left(a+\frac{d}{2}\right) - \cos\left(a+3\frac{d}{2}\right) \\
&\quad + \cos\left(a+3\frac{d}{2}\right) - \cdots + \cos\left(a+(2n-3)\frac{d}{2}\right) - \cos\left(a+(2n-1)\frac{d}{2}\right) \\
&= \cos\left(a-\frac{d}{2}\right) - \cos\left(a+(2n-1)\frac{d}{2}\right).
\end{aligned}
$$

   Thus, $\left|2\sin\frac{d}{2}\cdot x_n\right| = \left|\cos\left(a-\frac{d}{2}\right) - \cos\left(a+(2n-1)\frac{d}{2}\right)\right| \leq 2$, for all $n \geq 1$. In other words, for every $n \geq 1$, we have $|x_n| \leq |\operatorname{cosec}\frac{d}{2}|$. Therefore, the sequence is bounded whenever $d/2$ is not an integer multiple of $\pi$.

   Next, let us consider the case when $d/2 = k\pi$ for some integer $k$. In this case, we have $\sin(a+jd) = \sin(a+j\cdot 2k\pi) = \sin a$ for every integer $j$. Therefore $x_n = n\sin a$, for $n \geq 1$. So in this case, the sequence $x_n$ is bounded if only if $\sin a = 0$, i.e. if and only if $a = m\pi$ for some integer $m$.

2. *Suppose that* $\{x_n\}_{n\geq 1}$ *and* $\{y_n\}_{n\geq 1}$ *are two convergent sequences, with* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. *Determine (with proof/counter-example) whether the following statements are true or false:*

   (a) $\lim_{n\to\infty}(x_1 + \cdots + x_n) = \lim_{n\to\infty}(y_1 + \cdots + y_n)$.

   (b) $\lim_{n\to\infty}(x_n)^n = \lim_{n\to\infty}(y_n)^n$.

   *(Note, a statement is false if it fails to hold even for just one case.)*

Solution: Both the statements are **false** (i.e. they are not necessarily true). Counter-examples are given below.

(a) Take $x_n = 1/2^n$ and $y_n = 1/3^n$. (In fact you can take $x_n$ to be the $n$-th term of any convergent series and take $y_n = 0$.)

(b) Take $x_n = 2^{1/n}$ and $y_n = 3^{1/n}$. There are many other counter-examples too.

3. *Suppose that $x_n$ satisfies $x_{n+1} = \sqrt{6 + x_n}$ for every $n \geq 1$, and let $x_1 = \sqrt{6}$. Show that $x_n$ converges and also find the limit.*

   Solution: First note that $x_n > 0$ for all $n \geq 1$. Next, note that we can also give an upper bound. We shall induct on $n$ to show that $x_n < 3$ for all $n \geq 1$. Base case: $x_1 = \sqrt{6} < 3$. Inductive step: $x_{n+1} = \sqrt{6 + x_n} < \sqrt{6 + 3} = 3$.

   Thus, we have shown that $0 < x_n < 3$ holds for every $n \geq 1$. Next, we shall try to see whether the sequence is increasing or decreasing. We do a rough work:

$$x_{n+1} < x_n \iff \sqrt{6 + x_n} < x_n \iff 6 + x_n < x_n^2 \iff 0 < (x_n - 3)(x_n + 2).$$

   But, the last inequality is false. In fact, we have $(x_n - 3)(x_n + 2) < 0$ for each $n \geq 1$. Therefore, reversing the inequality sign in the above calculation, we arrive at $x_{n+1} > x_n$ for each $n \geq 1$.

   Thus $x_n$ is increasing and bounded above, hence convergent. Say $\lim_{n\to\infty} x_n = \ell$. Letting $n \to \infty$ in the recurrence $x_{n+1} = \sqrt{6 + x_n}$, we get $\ell = \sqrt{6 + \ell} \iff (\ell - 3)(\ell + 2) = 0 \iff \ell = 3$ or $-2$. Since $x_n > 0$ for all $n \geq 1$, the limit must be $\ell = 3$.

4. *Suppose $a$ is a positive real number. Define a sequence $\{x_n\}_{n\geq 1}$ by*

$$x_n = \frac{[a] + [2a] + \cdots + [na]}{n^2}, n \geq 1.$$

   *Prove that $\lim\limits_{n\to\infty} x_n$ exists and also find the limit. (Here $[t]$ denotes the greatest integer less than or equal to $t$.)*

   Solution: We shall give upper and lower bound and see whether we can apply Sandwich theorem. The most crucial bounds for $[x]$ is that: $x - 1 < [x] \leq x$ for every $x \in \mathbb{R}$. Using this, we get

$$\frac{n(n+1)}{2}a - n = \sum_{k=1}^{n}(ka - 1) < \sum_{k=1}^{n}[ka] \leq \sum_{k=1}^{n} ka = \frac{n(n+1)}{2}a.$$

Now, observe that

$$\lim_{n\to\infty} \frac{1}{n^2}\Big(\frac{n(n+1)}{2}a - n\Big) = \frac{a}{2} = \lim_{n\to\infty} \frac{1}{n^2}\frac{n(n+1)}{2}a.$$

So Sandwich theorem applies here and tells us that the given limit exists and equals $a/2$.