CHAPTER 1

Numbers and Functions

The subject of this course is "functions of one real variable" so we begin by wondering what a real number "really" is, and then, in the next section, what a function is.

1. What is a number?

1.1. Different kinds of numbers. The simplest numbers are the ***positive integers***

$$1, 2, 3, 4, \cdots$$

the number ***zero***

$$0,$$

and the ***negative integers***

$$\cdots, -4, -3, -2, -1.$$

Together these form the integers or "whole numbers."

Next, there are the numbers you get by dividing one whole number by another (nonzero) whole number. These are the so called fractions or ***rational numbers*** such as

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}, \cdots$$

or

$$-\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}, -\frac{2}{4}, -\frac{3}{4}, -\frac{4}{3}, \cdots$$

By definition, any whole number is a rational number (in particular zero is a rational number.)

You can add, subtract, multiply and divide any pair of rational numbers and the result will again be a rational number (provided you don't try to divide by zero).

One day in middle school you were told that there are other numbers besides the rational numbers, and the first example of such a number is the square root of two. It has been known ever since the time of the greeks that no rational number exists whose square is exactly 2, i.e. you can't find a fraction $\frac{m}{n}$ such that

$$\left(\frac{m}{n}\right)^2 = 2, \text{ i.e. } m^2 = 2n^2.$$

x	x^2
1.2	1.44
1.3	1.69
1.4	$1.96 < 2$
1.5	$2.25 > 2$
1.6	2.56

Nevertheless, if you compute x^2 for some values of x between 1 and 2, and check if you get more or less than 2, then it looks like there should be some number x between 1.4 and 1.5 whose square is exactly 2. So, we *assume* that there is such a number, and we call it the square root of 2, written as $\sqrt{2}$. This raises several questions. How do we know there really is a number between 1.4 and 1.5 for which $x^2 = 2$? How many other such numbers are we going to assume into existence? Do these new numbers obey the same algebra rules (like $a + b = b + a$) as the rational numbers? If we knew precisely what these numbers (like $\sqrt{2}$) were then we could perhaps answer such questions. It turns out to be rather difficult to give a precise description of what a number is, and in this course we won't try to get anywhere near the bottom of this issue. Instead we will think of numbers as "infinite decimal expansions" as follows.

One can represent certain fractions as decimal fractions, e.g.

$$\frac{279}{25} = \frac{1116}{100} = 11.16.$$

Not all fractions can be represented as decimal fractions. For instance, expanding $\frac{1}{3}$ into a decimal fraction leads to an unending decimal fraction

$$\frac{1}{3} = 0.333\,333\,333\,333\,333 \cdots$$

It is impossible to write the complete decimal expansion of $\frac{1}{3}$ because it contains infinitely many digits. But we can describe the expansion: each digit is a three. An electronic calculator, which always represents numbers as ***finite*** decimal numbers, can never hold the number $\frac{1}{3}$ exactly.

Every fraction can be written as a decimal fraction which may or may not be finite. If the decimal expansion doesn't end, then it must repeat. For instance,

$$\frac{1}{7} = 0.142857\,142857\,142857\,142857\,\ldots$$

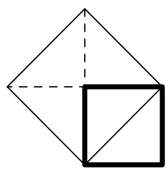
Conversely, any infinite repeating decimal expansion represents a rational number.

A ***real number*** is specified by a possibly unending decimal expansion. For instance,

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,376\,9\ldots$$

Of course you can never write *all* the digits in the decimal expansion, so you only write the first few digits and hide the others behind dots. To give a precise description of a real number (such as $\sqrt{2}$) you have to explain how you could *in principle* compute as many digits in the expansion as you would like. During the next three semesters of calculus we will not go into the details of how this should be done.

1.2. A reason to believe in $\sqrt{2}$. The Pythagorean theorem says that the hypotenuse of a right triangle with sides 1 and 1 must be a line segment of length $\sqrt{2}$. In middle or high school you learned something similar to the following geometric construction of a line segment whose length is $\sqrt{2}$. Take a square with side of length 1, and construct a new square one of whose sides is the diagonal of the first square. The figure you get consists of 5 triangles of equal area and by counting triangles you see that the larger square has exactly twice the area of the smaller square. Therefore the diagonal of the smaller square, being the side of the larger square, is $\sqrt{2}$ as long as the side of the smaller square.



Why are real numbers called real? All the numbers we will use in this first semester of calculus are "real numbers." At some point (in 2nd semester calculus) it becomes useful to assume that there is a number whose square is -1. No real number has this property since the square of any real number is positive, so it was decided to call this new imagined number "imaginary" and to refer to the numbers we already have (rationals, $\sqrt{2}$-like things) as "real."

1.3. The real number line and intervals. It is customary to visualize the real numbers as points on a straight line. We imagine a line, and choose one point on this line, which we call the ***origin***. We also decide which direction we call "left" and hence which we call "right." Some draw the number line vertically and use the words "up" and "down."

To plot any real number x one marks off a distance x from the origin, to the right (up) if $x > 0$, to the left (down) if $x < 0$.

The ***distance along the number line*** between two numbers x and y is $|x - y|$. In particular, the distance is never a negative number.

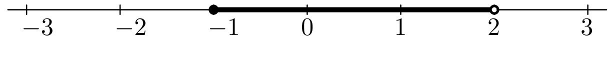


Figure 1. To draw the half open interval $[-1, 2)$ use a filled dot to mark the endpoint which is included and an open dot for an excluded endpoint.

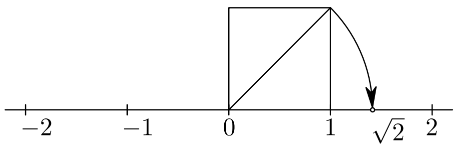


Figure 2. To find $\sqrt{2}$ on the real line you draw a square of sides 1 and drop the diagonal onto the real line.

Almost every equation involving variables x, y, etc. we write down in this course will be true for some values of x but not for others. In modern abstract mathematics a collection of real numbers (or any other kind of mathematical objects) is called a ***set***. Below are some examples of sets of real numbers. We will use the notation from these examples throughout this course.

The collection of all real numbers between two given real numbers form an interval. The following notation is used

- (a, b) is the set of all real numbers x which satisfy $a < x < b$.
- $[a, b)$ is the set of all real numbers x which satisfy $a \leq x < b$.
- $(a, b]$ is the set of all real numbers x which satisfy $a < x \leq b$.
- $[a, b]$ is the set of all real numbers x which satisfy $a \leq x \leq b$.

If the endpoint is not included then it may be ∞ or $-\infty$. E.g. $(-\infty, 2]$ is the interval of all real numbers (both positive and negative) which are ≤ 2.

1.4. Set notation. A common way of describing a set is to say it is the collection of all real numbers which satisfy a certain condition. One uses this notation

$$\mathcal{A} = \bigl\{x \mid x \text{ satisfies this or that condition}\bigr\}$$

Most of the time we will use upper case letters in a calligraphic font to denote sets. ($\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$, ...)

For instance, the interval (a, b) can be described as

$$(a, b) = \bigl\{x \mid a < x < b\bigr\}$$

The set

$$\mathcal{B} = \bigl\{x \mid x^2 - 1 > 0\bigr\}$$

consists of all real numbers x for which $x^2 - 1 > 0$, i.e. it consists of all real numbers x for which either $x > 1$ or $x < -1$ holds. This set consists of two parts: the interval $(-\infty, -1)$ and the interval $(1, \infty)$.

You can try to draw a set of real numbers by drawing the number line and coloring the points belonging to that set red, or by marking them in some other way.

Some sets can be very difficult to draw. For instance,

$$\mathcal{C} = \bigl\{x \mid x \text{ is a rational number}\bigr\}$$

can't be accurately drawn. In this course we will try to avoid such sets.

Sets can also contain just a few numbers, like

$$\mathcal{D} = \{1, 2, 3\}$$

which is the set containing the numbers one, two and three. Or the set

$$\mathcal{E} = \bigl\{x \mid x^3 - 4x^2 + 1 = 0\bigr\}$$

which consists of the solutions of the equation $x^3 - 4x^2 + 1 = 0$. (There are three of them, but it is not easy to give a formula for the solutions.)

If $\mathcal{A}$ and $\mathcal{B}$ are two sets then ***the union of*** $\mathcal{A}$ ***and*** $\mathcal{B}$ is the set which contains all numbers that belong either to $\mathcal{A}$ or to $\mathcal{B}$. The following notation is used

$$\mathcal{A} \cup \mathcal{B} = \bigl\{x \mid x \text{ belongs to } \mathcal{A} \text{ or to } \mathcal{B} \text{ or both.}\bigr\}$$

Similarly, the ***intersection of two sets*** $\mathcal{A}$ ***and*** $\mathcal{B}$ is the set of numbers which belong to both sets. This notation is used:

$$\mathcal{A} \cap \mathcal{B} = \{x \mid x \text{ belongs to both } \mathcal{A} \text{ and } \mathcal{B}.\}$$

2. Exercises

1. What is the 2007^{th} digit after the period in the expansion of $\frac{1}{7}$?

2. Which of the following fractions have finite decimal expansions?

$$a = \frac{2}{3}, \quad b = \frac{3}{25}, \quad c = \frac{276937}{15625}.$$

3. Draw the following sets of real numbers. Each of these sets is the union of one or more intervals. Find those intervals. Which of thee sets are finite?

$\mathcal{A} = \{x \mid x^2 - 3x + 2 \le 0\}$
$\mathcal{B} = \{x \mid x^2 - 3x + 2 \ge 0\}$
$\mathcal{C} = \{x \mid x^2 - 3x > 3\}$
$\mathcal{D} = \{x \mid x^2 - 5 > 2x\}$
$\mathcal{E} = \{t \mid t^2 - 3t + 2 \le 0\}$
$\mathcal{F} = \{\alpha \mid \alpha^2 - 3\alpha + 2 \ge 0\}$
$\mathcal{G} = (0, 1) \cup (5, 7]$
$\mathcal{H} = \big(\{1\} \cup \{2, 3\}\big) \cap (0, 2\sqrt{2})$
$\mathcal{Q} = \{\theta \mid \sin\theta = \frac{1}{2}\}$
$\mathcal{R} = \{\varphi \mid \cos\varphi > 0\}$

4. Suppose $\mathcal{A}$ and $\mathcal{B}$ are intervals. Is it always true that $\mathcal{A} \cap \mathcal{B}$ is an interval? How about $\mathcal{A} \cup \mathcal{B}$?

5. Consider the sets

$$\mathcal{M} = \{x \mid x > 0\} \text{ and } \mathcal{N} = \{y \mid y > 0\}.$$

Are these sets the same?

6. Group Problem.

Write the numbers

$$x = 0.3131313131\ldots, \quad y = 0.273273273273\ldots$$
$$\text{and } z = 0.215415415415415415\ldots$$

as fractions (i.e. write them as $\frac{m}{n}$, specifying m and n.)

(Hint: show that $100x = x + 31$. A similar trick works for y, but z is a little harder.)

7. Group Problem.

Is the number whose decimal expansion after the period consists only of nines, i.e.

$$x = 0.99999999999999999\ldots$$

an integer?

3. Functions

Wherein we meet the main characters of this semester

3.1. Definition. To specify a ***function*** f you must

(1) give a ***rule*** which tells you how to compute the value $f(x)$ of the function for a given real number x, and:
(2) say for which real numbers x the rule may be applied.

The set of numbers for which a function is defined is called its ***domain***. The set of all possible numbers $f(x)$ as x runs over the domain is called the ***range*** of the function. The rule must be ***unambiguous:*** the same $x$must always lead to the same $f(x)$.

For instance, one can define a function f by putting $f(x) = \sqrt{x}$ for all $x \ge 0$. Here the rule defining f is "take the square root of whatever number you're given", and the function f will accept all nonnegative real numbers.

The rule which specifies a function can come in many different forms. Most often it is a formula, as in the square root example of the previous paragraph. Sometimes you need a few formulas, as in

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases} \qquad \text{domain of } g = \text{all real numbers.}$$

Functions which are defined by different formulas on different intervals are sometimes called ***piecewise defined functions.***

3.2. Graphing a function. You get the ***graph of a function*** f by drawing all points whose coordinates are (x, y) where x must be in the domain of f and $y = f(x)$.

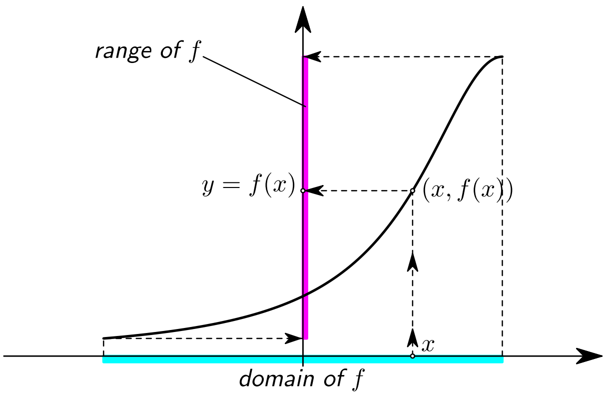


Figure 3. The graph of a function f. The domain of f consists of all x values at which the function is defined, and the range consists of all possible values f can have.

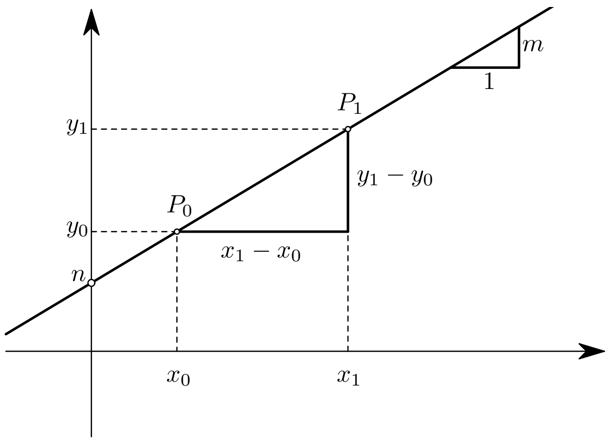


Figure 4. A straight line and its slope. The line is the graph of $f(x) = mx + n$. It intersects the y-axis at height n, and the ratio between the amounts by which y and x increase as you move from one point to another on the line is $\frac{y_1 - y_0}{x_1 - x_0} = m$.

3.3. Linear functions. A function which is given by the formula

$$f(x) = mx + n$$

where m and n are constants is called a ***linear function***. Its graph is a straight line. The constants m and n are the ***slope*** and ***y-intercept*** of the line. Conversely, any straight line which is not vertical (i.e. not parallel to the y-axis) is the graph of a linear function. If you know two points (x_0, y_0) and (x_1, y_1) on the line, then then one can compute the slope m from the "rise-over-run" formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

This formula actually contains a theorem from Euclidean geometry, namely it says that the ratio $(y_1 - y_0) : (x_1 - x_0)$ is the same for every pair of points (x_0, y_0) and (x_1, y_1) that you could pick on the line.

3.4. Domain and "biggest possible domain. " In this course we will usually not be careful about specifying the domain of the function. When this happens the domain is understood to be the set of all x for which the rule which tells you how to compute $f(x)$ is meaningful. For instance, if we say that h is the function

$$h(x) = \sqrt{x}$$

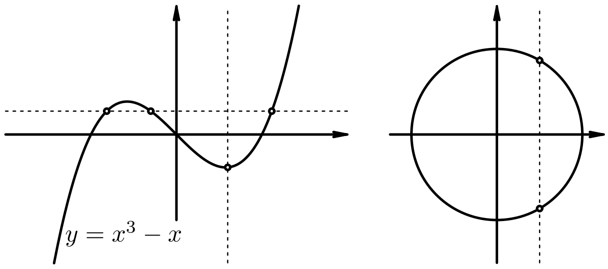


Figure 5. The graph of $y = x^3 - x$ fails the "horizontal line test," but it passes the "vertical line test." The circle fails both tests.

then the domain of h is understood to be the set of all nonnegative real numbers

$$\text{domain of } h = [0, \infty)$$

since $\sqrt{x}$ is well-defined for all $x \geq 0$ and undefined for $x < 0$.

A systematic way of finding the domain and range of a function for which you are only given a formula is as follows:

- The domain of f consists of all x for which $f(x)$ is well-defined ("makes sense")
- The range of f consists of all y for which you can solve the equation $f(x) = y$.

3.5. Example – find the domain and range of $f(x) = 1/x^2$. The expression $1/x^2$ can be computed for all real numbers x except $x = 0$ since this leads to division by zero. Hence the domain of the function $f(x) = 1/x^2$ is

$$\text{"all real numbers except 0"} = \big\{x \mid x \neq 0\big\} = (-\infty, 0) \cup (0, \infty).$$

To find the range we ask "for which y can we solve the equation $y = f(x)$ for x," i.e. we for which y can you solve $y = 1/x^2$ for x?

If $y = 1/x^2$ then we must have $x^2 = 1/y$, so first of all, since we have to divide by y, y can't be zero. Furthermore, $1/y = x^2$ says that y must be positive. On the other hand, if $y > 0$ then $y = 1/x^2$ has a solution (in fact two solutions), namely $x = \pm 1/\sqrt{y}$. This shows that the range of f is

$$\text{"all positive real numbers"} = \{x \mid x > 0\} = (0, \infty).$$

3.6. Functions in "real life." One can describe the motion of an object using a function. If some object is moving along a straight line, then you can define the following function: Let $x(t)$ be the distance from the object to a fixed marker on the line, at the time t. Here the domain of the function is the set of all times t for which we know the position of the object, and the rule is

Given t, measure the distance between the object and the marker at time t.

There are many examples of this kind. For instance, a biologist could describe the growth of a cell by defining $m(t)$ to be the mass of the cell at time t (measured since the birth of the cell). Here the domain is the interval $[0, T]$, where T is the life time of the cell, and the rule that describes the function is

Given t, weigh the cell at time t.

3.7. The Vertical Line Property. Generally speaking graphs of functions are curves in the plane but they distinguish themselves from arbitrary curves by the way they intersect vertical lines: ***The graph of a function cannot intersect a vertical line "$x =$ constant" in more than one point.*** The reason why this is true is very simple: if two points lie on a vertical line, then they have the same x coordinate, so if they also lie on the graph of a function f, then their y-coordinates must also be equal, namely $f(x)$.

3.8. Examples. The graph of $f(x) = x^3 - x$ "goes up and down," and, even though it intersects several horizontal lines in more than one point, it intersects ***every*** vertical line in exactly one point.

The collection of points determined by the equation $x^2 + y^2 = 1$ is a circle. It is not the graph of a function since the vertical line $x = 0$ (the y-axis) intersects the graph in two points $P_1(0,1)$ and $P_2(0,-1)$. See Figure 6.

4. Inverse functions and Implicit functions

For many functions the rule which tells you how to compute it is not an explicit formula, but instead an equation which you still must solve. A function which is defined in this way is called an "implicit function."

4.1. Example. One can define a function f by saying that for each x the value of $f(x)$ is the solution y of the equation

$$x^2 + 2y - 3 = 0.$$

In this example you can solve the equation for y,

$$y = \frac{3 - x^2}{2}.$$

Thus we see that the function we have defined is $f(x) = (3 - x^2)/2$.

Here we have two definitions of the same function, namely

(i) "$y = f(x)$ is defined by $x^2 + 2y - 3 = 0$," and
(ii) "f is defined by $f(x) = (3 - x^2)/2$."

The first definition is the implicit definition, the second is explicit. You see that with an "implicit function" it isn't the function itself, but rather the way it was defined that's implicit.

4.2. Another example: domain of an implicitly defined function. Define g by saying that for any x the value $y = g(x)$ is the solution of

$$x^2 + xy - 3 = 0.$$

Just as in the previous example one can then solve for y, and one finds that

$$g(x) = y = \frac{3 - x^2}{x}.$$

Unlike the previous example this formula does not make sense when $x = 0$, and indeed, for $x = 0$ our rule for g says that $g(0) = y$ is the solution of

$$0^2 + 0 \cdot y - 3 = 0, \text{ i.e. } y \text{ is the solution of } 3 = 0.$$

That equation has no solution and hence $x = 0$ does not belong to the domain of our function g.

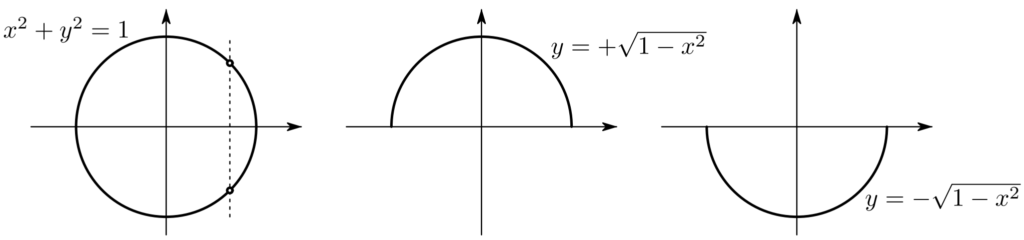


Figure 6. The circle determined by $x^2 + y^2 = 1$ is not the graph of a function, but it contains the graphs of the two functions $h_1(x) = \sqrt{1 - x^2}$ and $h_2(x) = -\sqrt{1 - x^2}$.

4.3. Example: the equation alone does not determine the function. Define $y = h(x)$ to be the solution of

$$x^2 + y^2 = 1.$$

If $x > 1$ or $x < -1$ then $x^2 > 1$ and there is no solution, so $h(x)$ is at most defined when $-1 \le x \le 1$. But when $-1 < x < 1$ there is another problem: not only does the equation have a solution, but it even has two solutions:

$$x^2 + y^2 = 1 \iff y = \sqrt{1-x^2} \text{ or } y = -\sqrt{1-x^2}.$$

The rule which defines a function must be unambiguous, and since we have not specified which of these two solutions is $h(x)$ the function is not defined for $-1 < x < 1$.

One can fix this by making a choice, but there are many possible choices. Here are three possibilities:

$$\begin{aligned} h_1(x) &= \text{the nonnegative solution } y \text{ of } x^2 + y^2 = 1 \\ h_2(x) &= \text{the nonpositive solution } y \text{ of } x^2 + y^2 = 1 \\ h_3(x) &= \begin{cases} h_1(x) & \text{when } x < 0 \\ h_2(x) & \text{when } x \ge 0 \end{cases} \end{aligned}$$

4.4. Why use implicit functions? In all the examples we have done so far we could replace the implicit description of the function with an explicit formula. This is not always possible or if it is possible the implicit description is much simpler than the explicit formula. For instance, you can define a function f by saying that $y = f(x)$ if and only if

$$y^3 + 3y + 2x = 0. \tag{1}$$

This means that the recipe for computing $f(x)$ for any given x is "solve the equation $y^3 + 3y + 2x = 0$." E.g. to compute $f(0)$ you set $x = 0$ and solve $y^3 + 3y = 0$. The only solution is $y = 0$, so $f(0) = 0$. To compute $f(1)$ you have to solve $y^3 + 3y + 2 \cdot 1 = 0$, and if you're lucky you see that $y = -1$ is the solution, and $f(1) = -1$.

In general, no matter what x is, the equation (1) turns out to have exactly one solution y (which depends on x, this is how you get the function f). Solving (1) is not easy. In the early 1500s Cardano and Tartaglia discovered a formula[1] for the solution. Here it is:

$$y = f(x) = \sqrt[3]{-x + \sqrt{1+x^2}} - \sqrt[3]{x + \sqrt{1+x^2}}.$$

The implicit description looks a lot simpler, and when we try to differentiate this function later on, it will be much easier to use "implicit differentiation" than to use the Cardano-Tartaglia formula directly.

4.5. Inverse functions. If you have a function f, then you can try to define a new function f^{-1}, the so-called ***inverse function of*** f**,** by the following prescription:

(2) For any given x we say that $y = f^{-1}(x)$ if y is the solution to the equation $f(y) = x$.

So to find $y = f^{-1}(x)$ you solve the equation $x = f(y)$. If this is to define a function then the prescription (2) must be unambiguous and the equation $f(y) = x$ has to have a solution and cannot have more than one solution.

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[1] To see the solution and its history visit

http://www.gap-system.org/~history/HistTopics/Quadratic_etc_equations.html

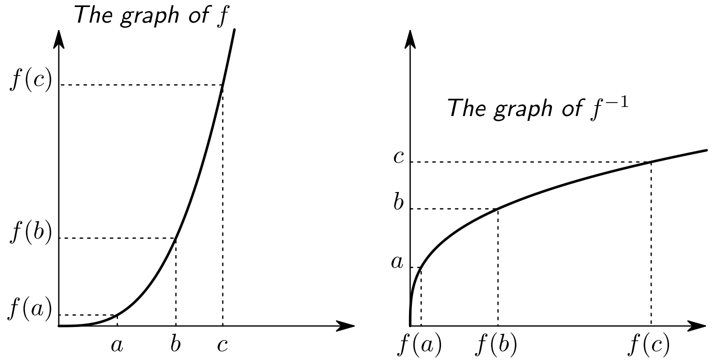

Figure 7. The graph of a function and its inverse are mirror images of each other.

4.6. Examples. Consider the function f with $f(x) = 2x + 3$. Then the equation $f(y) = x$ works out to be

$$2y + 3 = x$$

and this has the solution

$$y = \frac{x-3}{2}.$$

So $f^{-1}(x)$ is defined for all x, and it is given by $f^{-1}(x) = (x-3)/2$.

Next we consider the function $g(x) = x^2$ with domain all positive real numbers. To see for which x the inverse $g^{-1}(x)$ is defined we try to solve the equation $g(y) = x$, i.e. we try to solve $y^2 = x$. If $x < 0$ then this equation has no solutions since $y^{\geq}0$ for all y. But if $x \geq 0$ then $y^{=}x$ does have a solution, namely $y = \sqrt{x}$.

So we see that $g^{-1}(x)$ is defined for all nonnegative real numbers x, and that it is given by $g^{-1}(x) = \sqrt{x}$.

4.7. Inverse trigonometric functions. The familiar trigonometric functions Sine, Cosine and Tangent have inverses which are called arcsine, arccosine and arctangent.

$y = f(x)$		$x = f^{-1}(y)$	
$y = \sin x$	$(-\pi/2 \leq x \leq \pi/2)$	$x = \arcsin(y)$	$(-1 \leq y \leq 1)$
$y = \cos x$	$(0 \leq x \leq \pi)$	$x = \arccos(y)$	$(-1 \leq y \leq 1)$
$y = \tan x$	$(-\pi/2 < x < \pi/2)$	$x = \arctan(y)$	

The notations $\arcsin y = \sin^{-1} y$, $\arccos x = \cos^{-1} x$, and $\arctan u = \tan^{-1} u$ are also commonly used for the inverse trigonometric functions. We will avoid the $\sin^{-1} y$ notation because it is ambiguous. Namely, everybody writes the square of $\sin y$ as

$$\bigl(\sin y\bigr)^2 = \sin^2 y.$$

Replacing the 2's by -1's would lead to

$$\arcsin y = \sin^{-1} y \overset{?!?}{=} \bigl(\sin y\bigr)^{-1} = \frac{1}{\sin y}, \quad \textbf{which is not true!}$$

5. Exercises

8. The functions f and g are defined by

$$f(x) = x^2 \text{ and } g(s) = s^2.$$

Are f and g the same functions or are they different?

9. Find a formula for the function f which is defined by

$$y = f(x) \iff x^2y + y = 7.$$

What is the domain of f?

10. Find a formula for the function f which is defined by

$$y = f(x) \iff x^2y - y = 6.$$

What is the domain of f?

11. Let f be the function defined by $y = f(x) \iff y$ is the largest solution of

$$y^2 = 3x^2 - 2xy.$$

Find a formula for f. What are the domain and range of f?

12. Find a formula for the function f which is defined by

$$y = f(x) \iff 2x + 2xy + y^2 = 5 \text{ and } y > -x.$$

Find the domain of f.

13. Use a calculator to compute $f(1.2)$ in three decimals where f is the implicitly defined function from §4.4. (There are (at least) two different ways of finding $f(1.2)$)

14. Group Problem.

(a) True or false:
for all x one has $\sin(\arcsin x) = x$?

(b) True or false:
for all x one has $\arcsin(\sin x) = x$?

15. On a graphing calculator plot the graphs of the following functions, and explain the results. (Hint: first do the previous exercise.)

$$\begin{aligned}
f(x) &= \arcsin(\sin x), \quad -2\pi \le x \le 2\pi \\
g(x) &= \arcsin(x) + \arccos(x), \quad 0 \le x \le 1 \\
h(x) &= \arctan \frac{\sin x}{\cos x}, \quad |x| < \pi/2 \\
k(x) &= \arctan \frac{\cos x}{\sin x}, \quad |x| < \pi/2 \\
l(x) &= \arcsin(\cos x), \quad -\pi \le x \le \pi \\
m(x) &= \cos(\arcsin x), \quad -1 \le x \le 1
\end{aligned}$$

16. Find the inverse of the function f which is given by $f(x) = \sin x$ and **whose domain is** $\pi \le x \le 2\pi$. Sketch the graphs of both f and f^{-1}.

17. Find a number a such that the function $f(x) = \sin(x + \pi/4)$ with domain $a \le x \le a + \pi$ has an inverse. Give a formula for $f^{-1}(x)$ using the arcsine function.

18. Draw the graph of the function h_3 from §4.3.

19. A function f is given which satisfies

$$f(2x + 3) = x^2$$

for all real numbers x.

Compute

(a) $f(0)$ (b) $f(3)$ (c) $f(x)$
(d) $f(y)$ (e) $f(f(2))$

where x and y are arbitrary real numbers.

What are the range and domain of f?

20. A function f is given which satisfies

$$f\Big(\frac{1}{x+1}\Big) = 2x - 12.$$

for all real numbers x.

Compute

(a) $f(1)$ (b) $f(0)$ (c) $f(x)$
(d) $f(t)$ (e) $f(f(2))$

where x and t are arbitrary real numbers.

What are the range and domain of f?

21. Does there exist a function f which satisfies

$$f(x^2) = x + 1$$

for **all** real numbers x?

* * *

The following exercises review precalculus material involving quadratic expressions $ax^2 + bx + c$ in one way or another.

22. Explain how you "complete the square" in a quadratic expression like $ax^2 + bx$.

23. Find the range of the following functions:

$$\begin{aligned}
f(x) &= 2x^2 + 3 \\
g(x) &= -2x^2 + 4x \\
h(x) &= 4x + x^2 \\
k(x) &= 4\sin x + \sin^2 x \\
\ell(x) &= 1/(1 + x^2) \\
m(x) &= 1/(3 + 2x + x^2).
\end{aligned}$$

24. Group Problem.

For each real number a we define a line ℓ_a with equation $y = ax + a^2$.

(a) Draw the lines corresponding to $a = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.

(b) Does the point with coordinates $(3, 2)$ lie on one or more of the lines ℓ_a (where a can be any number, not just the five values from part (a))? If so, for which values of a does $(3, 2)$ lie on ℓ_a?

(c) Which points in the plane lie on at least one of the lines ℓ_a?.

25. For which values of m and n does the graph of $f(x) = mx + n$ intersect the graph of $g(x) = 1/x$ in exactly one point and also contain the point $(-1, 1)$?

26. For which values of m and n does the graph of $f(x) = mx + n$ **not** intersect the graph of $g(x) = 1/x$?