Some Problems on Functions

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1. Suppose $f(x) = \dfrac{9^x}{9^x + 3}$. Find the value of

$$f\Big(\frac{1}{2019}\Big) + f\Big(\frac{2}{2019}\Big) + f\Big(\frac{3}{2019}\Big) + \cdots + f\Big(\frac{2018}{2019}\Big).$$

2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$. Is f one-one? Is it onto? If it is not onto, can you find the range?

3. Suppose that $f : A \to B$ and $g : B \to C$ are functions such that g is one-one and $g \circ f : A \to C$ is bijective. Determine whether the following are necessarily true: (i) f is one-one, (ii) g is onto, (iii) f is onto.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies $f(f(f(x))) = x$ for every $x \in \mathbb{R}$. Show that (i) f is bijective and (ii) f can not be strictly decreasing. Is it necessary that $f(x) = x$ for all $x \in \mathbb{R}$?

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfies $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$.

6. Let $\mathcal{S}$ be the set of all points in a plane. Suppose $f : \mathcal{S} \to \mathbb{R}$ is a function such that for every square $ABCD$ it holds that $f(A) + f(B) + f(C) + f(D) = 0$. Prove that, $f(P) = 0$ for every $P \in \mathcal{S}$. Note, degenerate square (square of side length 0) is not allowed.

7. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is bijective. Show that there exists $a, b, c \in \mathbb{N}$ which are in arithmetic progression and $f(a) < f(b) < f(c)$ holds.

8. Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is onto and g is one-one. Furthermore, assume that $f(n) \geq g(n)$ holds for every $n \in \mathbb{N}$. Prove that $f = g$.

9. Determine whether there exists a one-one function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2) - f(x)^2 \geq \frac{1}{4}$$

for every $x \in \mathbb{R}$.

10. Suppose $f : \mathbb{N} \to \mathbb{N}$ has the property that for every $n \in \mathbb{N}$,

$$f(1) + f(2) + \cdots + f(n) = c_n^3 \leq n^3$$

where $c_n \in \mathbb{N}$. Find $f(n)$.

11. Show that for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor .$$

(Hint: One approach is to use periodicity.)

12. Let α, β be positive irrational numbers such that $1/\alpha + 1/\beta = 1$. Show that the sequences $f(n) = \lfloor \alpha n \rfloor$ and $g(n) = \lfloor \beta n \rfloor$ $(n \in \mathbb{N})$ are disjoint and their union is $\mathbb{N}$. (In other words, show that $\text{Range}(f) \cap \text{Range}(g) = \phi$ and $\text{Range}(f) \cup \text{Range}(g) = \mathbb{N}$.)

13. $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(m + n) \leq f(m) + f(n)$ for every $m, n \in \mathbb{N}$. Show that,

$$f(1) + \frac{f(2)}{2} + \frac{f(3)}{3} + \cdots + \frac{f(n)}{n} \geq f(n)$$

holds for every $n \in \mathbb{N}$.

14. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(n)+1$ for every $n \in \mathbb{N}$. Find the maximum value of $f(n)$ when $1 \leq n \leq 2019$. (Hint: Think about representing n in binary.)

Hints/Answers

1. Show that $f(x) + f(-x) = 1$ for every $x \in \mathbb{R}$.

2. Note that for $x \leq 0$, $f(x) = 0$. And for $x > 0$, $f(x) = (e^x - e^{-x})/(e^x + e^{-x})$. Clearly, f is neither one-one, nor onto. The range of $f(x)$ is $[0, 1)$.

3. (i) $f(x) = f(y) \implies g(f(x)) = g(f(y)) \implies x = y$.

   (ii) Take any $c \in C$. Since $g \circ f$ is onto, there exists $a \in A$ such that $g(f(a)) = c$. Hence we get $b = f(a) \in B$ such that $g(b) = c$.

   (iii) Take any $b \in B$. Then $c = g(b) \in C$. Now, since $g \circ f$ is onto, there exists $a \in A$ such that $g(f(a)) = c$. Hence we get $g(f(a)) = g(b) \implies f(a) = b$ (as g is one-one).

4. It is easy to show that f is one-one and can not be strictly decreasing. For the last part, the answer is 'No'. Define $f(x) = 1/(1-x)$ for $x \neq 0, 1$. And set $f(0) = 0, f(1) = 1$. Check that $f(f(f(x))) = x$ holds for all x.

5. Fix x, y. Let $z = (x+y)/2$. Observe that $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq (x-z)^2 + (z-y)^2 = (x-y)^2/2$. This holds for every x, y. Prove (by induction) that for any x, y and for every $n \geq 1$, it holds that $|f(x) - f(y)| \leq (x-y)^2/2^n$. Now fix x, y and let $n \to \infty$ which gives us $|f(x) - f(y)| \leq 0 \implies f(x) = f(y)$. Thus, we get $f(x) = f(y)$ for every x, y; which means f is a constant function.

6. Fix any point P. Consider a 2×2 grid whose center is P. Sum the value of f on the vertices of each small square. Observe that there are *two* more squares to be considered.

7. Since f is surjective, there exists $a \in \mathbb{N}$ such that $f(a) = 1$. Then, since f is injective, we have $f(a) = 1 < f(n)$ for every $n \neq a$. Consider $a+1, a+2, a+4, a+8, \cdots$. Convince yourself that we can't have $f(b) > f(c)$ for all b, c which are consecutive terms of this sequence (i.e. we can't have $f(a+2^{k-1}) > f(a+2^k)$ for all $k \geq 1$). Hence, there must exist some $k \geq 1$ such that $f(a+2^{k-1}) < f(a+2^k)$. Take $b = a+2^{k-1}$ and $c = a+2^k$. Then, a, b, c are in arithmetic progression and $f(a) < f(b) < f(c)$.

8. There exists $n_1 \in \mathbb{N}$ such that $f(n_1) = 1$. Then, $1 = f(n_1) \geq g(n_1) \implies g(n_1) = 1$. Again, there exists $n_2 \in \mathbb{N}$ such that $f(n_2) = 2$. Hence, $2 = f(n_2) \geq g(n_2) \implies g(n_2) = 1$ or 2. But g is one-one and $n_2 \neq n_1$, so we must have $g(n_2) = 2$. In this way, for every $k \in \mathbb{N}$, there exits $n_k \in \mathbb{N}$ such that $f(n_k) = k$, and we prove that $g(n_k) = k$ by induction on k. Therefore, we have $f(n_k) = g(n_k) = k$ for every $k \in \mathbb{N}$. [But it is not sufficient to tell that $f = g$, because $\{n_k : k \geq 1\}$ might be just a proper subset of $\mathbb{N}$. To complete the proof, we have to take any arbitrary $m \in \mathbb{N}$ and show that $f(m) = g(m)$.] Fix any $m \in \mathbb{N}$ and call $g(m) = k$. We have $k = f(n_k) = g(n_k)$. Hence, $g(m) = k = g(n_k) \implies m = n_k$. Therefore, $f(m) = f(n_k) = g(n_k) = g(m)$.

9. Put $x = 0$ first and get $(f(0) - 1/2)^2 \leq 0 \implies f(0) = 1/2$. Now put $x = 1$ and get $f(1) = 1/2 = f(0)$, which contradicts that f is one-one.

10. Show $c_n = n$ by inducting on n. (Clearly this holds for $n = 1$. And if this holds for $n = k-1$, then we have $(k-1)^3 = c_{k-1} < c_k \leq k^3 \implies c_k = k^3$.) Hence

deduce that $f(n) = n^3 - (n-1)^3$ for every $n \in \mathbb{N}$.

11. Fix $n \in \mathbb{N}$. Consider $f(x) = \text{LHS} - \text{RHS}$. Show that $f(x + 1/n) = f(x)$ for every $x \in \mathbb{R}$. Now show that $f(x) = 0$ for $x \in [0, 1/n)$. Convince yourself that showing these two results completes the proof.

12. First we prove that they are disjoint. Let, if possible, $\lfloor \alpha m \rfloor = \lfloor \beta n \rfloor = q$. Then, $q < \alpha m, \beta n < n + 1$. Now show that this brings a contradiction. Next, we need to show that their union is $\mathbb{N}$. Observe that we must have $1 < \alpha, \beta < 2$. Hence, the intervals $[\alpha m, \alpha(m+1)]$ have length greater than 1 but not more than 2. So if some $k \in \mathbb{N}$ is missing from both the sequences, there must exists $q > 0$ such that $\alpha m < q < q + 1 < \alpha(m+1)$ and $\beta n < q < q + 1 < \beta(n+1)$. Deduce that $\frac{m+n}{q} < 1 < \frac{m+n+2}{q+1} \implies m + n < q < q + 1 < m + n + 2$ which is a contradiction.

13. We induct on n. The case $n = 1$ is trivially true. Suppose the assertion is true for all $n \leq k$. Then we have

$$f(1) \geq f(1),\ f(1) + \frac{f(2)}{2} \geq f(2),\ \cdots,\ f(1) + \frac{f(2)}{2} + \frac{f(3)}{3} + \cdots + \frac{f(k)}{k} \geq f(k).$$

Adding these inequalities altogether, we get

$$kf(1) + (k-1)\frac{f(2)}{2} + \cdots + \frac{f(k)}{k} \geq f(1) + f(2) + \cdots + f(k)$$

Next, adding $(f(1) + f(2) + \cdots + f(k))$ to both sides, we get

$$(k+1)\left(f(1) + \frac{f(2)}{2} + \cdots + \frac{f(k)}{k}\right) \geq \sum_{i=1}^{k} f(i) + f(k+1-i) \qquad (*)$$

Now, by using the given condition on f, we get, $f(i) + f(k+1-i) \geq f(i + k + 1 - i) = f(k+1)$ for each $i = 1, 2, \ldots k$. This combines with $(*)$ to give us,

$$f(1) + \frac{f(2)}{2} + \cdots + \frac{f(k)}{k} \geq \frac{k}{k+1} f(k+1) = f(k+1) - \frac{f(k+1)}{k+1}$$

This closes our induction.

14. Show that $f(n)$ equals the number of 1's in the binary representation of n. You can prove it by induction on the number of digits (in base 2) of n.