Ramanujan School of Mathematics

Class Test on Calculus, August 2019

Aditya Ghosh

Maximum marks: 40   Time: 1 hr 30 min.

Attempt as many as you can. Answers without proper explanations will fetch zero. You can use any result proved in the class, but you have to state it properly.

1. (10 marks) Start with an equilateral triangle with unit side length. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle. Repeat last step with each of the remaining smaller triangles.

Step 1   Step 2   Step 3   Step 4   Step 5
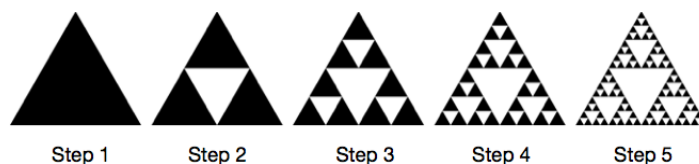
   Denote by $P(n)$ and $A(n)$ the perimeter and area of the existing portion of the triangle at the n-th step, e.g. $P(2) = 9/2$ unit and $A(2) = 3\sqrt{3}/16$ sq.unit. Find $\lim\limits_{n\to\infty} P(n)$ and $\lim\limits_{n\to\infty} A(n)$. Are you surprised?

2. (10 marks) Let x_n be a sequence of integers such that $x_{k+1} \neq x_k$ holds for every $k \geq 1$. Show that x_n can not be a Cauchy sequence. Is it possible that x_n has a convergent subsequence?

3. (10 marks) Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous. Show that there exists $a, b \in \mathbb{R}$ such that $b - a = 1$ and $f(b) - f(a) = \frac{1}{2}(f(2) - f(0))$.

4. (15 marks) Suppose that $f, g : [0, 1] \to [0, 1]$ are continuous functions such that $f(g(x)) = g(f(x))$ holds for every $x \in [0, 1]$.

   (a) (5 marks) Show that there exists $b \in [0, 1]$ such that $f(b) = b$.

   (b) (10 marks) Show that there exists $c \in [0, 1]$ such that $f(c) = g(c)$.

   [Hint: Construct a sequence by $a_{n+1} = g(a_n)$ and choose the first term suitably.]

Do not cheat to yourself. All the best!

Contact: +91-9831935258