Limits of Functions

In this chapter, we define limits of functions and describe their properties.

6.1. Limits

We begin with the ϵ-δ definition of the limit of a function.

Definition 6.1. Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

$$\lim_{x \to c} f(x) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

We also denote limits by the 'arrow' notation $f(x) \to L$ as $x \to c$, and often leave it to be implicitly understood that $x \in A$ is restricted to the domain of f. Note that it follows directly from the definition that

$$\lim_{x \to c} f(x) = L \quad \text{if and only if} \quad \lim_{x \to c} |f(x) - L| = 0.$$

In defining a limit as $x \to c$, we do not consider what happens when $x = c$, and a function needn't be defined at c for its limit to exist. This is the case, for example, when we define the derivative of a function as a limit of its difference quotients. Moreover, even if a function is defined at c and its limit as $x \to c$ exists, the value of the function need not equal the limit. In fact, the condition that $\lim_{x \to c} f(x) = f(c)$ defines the continuity of f at c. We study continuous functions in Chapter 7.

Example 6.2. Let $A = [0, \infty) \setminus \{9\}$ and define $f : A \to \mathbb{R}$ by

$$f(x) = \frac{x - 9}{\sqrt{x} - 3}.$$

We claim that

$$\lim_{x\to 9} f(x) = 6.$$

To prove this, let $\epsilon > 0$ be given. If $x \in A$, then $\sqrt{x} - 3 \neq 0$, and dividing this factor into the numerator we get $f(x) = \sqrt{x} + 3$. It follows that

$$|f(x) - 6| = \left|\sqrt{x} - 3\right| = \left|\frac{x-9}{\sqrt{x}+3}\right| \leq \frac{1}{3}|x - 9|.$$

Thus, if $\delta = 3\epsilon$, then $x \in A$ and $|x - 9| < \delta$ implies that $|f(x) - 6| < \epsilon$.

Like the limits of sequences, limits of functions are unique.

Proposition 6.3. The limit of a function is unique if it exists.

Proof. Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is an accumulation point of $A \subset \mathbb{R}$. Assume that

$$\lim_{x\to c} f(x) = L_1, \qquad \lim_{x\to c} f(x) = L_2$$

where $L_1, L_2 \in \mathbb{R}$. For every $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L_1| < \epsilon/2,$$
$$0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |f(x) - L_2| < \epsilon/2.$$

Let $\delta = \min(\delta_1, \delta_2) > 0$. Then, since c is an accumulation point of A, there exists $x \in A$ such that $0 < |x - c| < \delta$. It follows that

$$|L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon.$$

Since this holds for arbitrary $\epsilon > 0$, we must have $L_1 = L_2$. □

Note that in this proof we used the requirement in the definition of a limit that c is an accumulation point of A. The limit definition would be vacuous if it was applied to a non-accumulation point, and in that case every $L \in \mathbb{R}$ would be a limit.

We can rephrase the ϵ-δ definition of limits in terms of neighborhoods. Recall from Definition 5.6 that a set $V \subset \mathbb{R}$ is a neighborhood of $c \in \mathbb{R}$ if $V \supset (c - \delta, c + \delta)$ for some $\delta > 0$, and $(c - \delta, c + \delta)$ is called a δ-neighborhood of c.

Definition 6.4. A set $U \subset \mathbb{R}$ is a punctured (or deleted) neighborhood of $c \in \mathbb{R}$ if $U \supset (c - \delta, c) \cup (c, c + \delta)$ for some $\delta > 0$. The set $(c - \delta, c) \cup (c, c + \delta)$ is called a punctured (or deleted) δ-neighborhood of c.

That is, a punctured neighborhood of c is a neighborhood of c with the point c itself removed.

Definition 6.5. Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

$$\lim_{x\to c} f(x) = L$$

if and only if for every neighborhood V of L, there is a punctured neighborhood U of c such that

$$x \in A \cap U \text{ implies that } f(x) \in V.$$

This is essentially a rewording of the ϵ-δ definition. If Definition 6.1 holds and V is a neighborhood of L, then V contains an ϵ-neighborhood of L, so there is a punctured δ-neighborhood U of c such that f maps $U \cap A$ into V, which verifies Definition 6.5. Conversely, if Definition 6.5 holds and $\epsilon > 0$, then $V = (L-\epsilon, L+\epsilon)$ is a neighborhood of L, so there is a punctured neighborhood U of c such that f maps $U \cap A$ into V, and U contains a punctured δ-neighborhood of c, which verifies Definition 6.1.

The next theorem gives an equivalent sequential characterization of the limit.

Theorem 6.6. Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{n \to \infty} f(x_n) = L.$$

for every sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} x_n = c.$$

Proof. First assume that the limit exists and is equal to L. Suppose that (x_n) is any sequence in A with $x_n \neq c$ that converges to c, and let $\epsilon > 0$ be given. From Definition 6.1, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$, and since $x_n \to c$ there exists $N \in \mathbb{N}$ such that $0 < |x_n - c| < \delta$ for all $n > N$. It follows that $|f(x_n) - L| < \epsilon$ whenever $n > N$, so $f(x_n) \to L$ as $n \to \infty$.

To prove the converse, assume that the limit does not exist or is not equal to L. Then there is an $\epsilon_0 > 0$ such that for every $\delta > 0$ there is a point $x \in A$ with $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon_0$. Therefore, for every $n \in \mathbb{N}$ there is an $x_n \in A$ such that

$$0 < |x_n - c| < \frac{1}{n}, \qquad |f(x_n) - L| \geq \epsilon_0.$$

It follows that $x_n \neq c$ and $x_n \to c$, but $f(x_n) \not\to L$, so the sequential condition does not hold. This proves the result. □

A non-existence proof for a limit directly from Definition 6.1 is often awkward. (One has to show that for every $L \in \mathbb{R}$ there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon_0$.) The previous theorem gives a convenient way to show that a limit of a function does not exist.

Corollary 6.7. Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is an accumulation point of A. Then $\lim_{x \to c} f(x)$ does not exist if either of the following conditions holds:

(1) There are sequences (x_n), (y_n) in A with $x_n, y_n \neq c$ such that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = c, \quad \text{but} \quad \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n).$$

(2) There is a sequence (x_n) in A with $x_n \neq c$ such that $\lim_{n \to \infty} x_n = c$ but the sequence $(f(x_n))$ diverges.

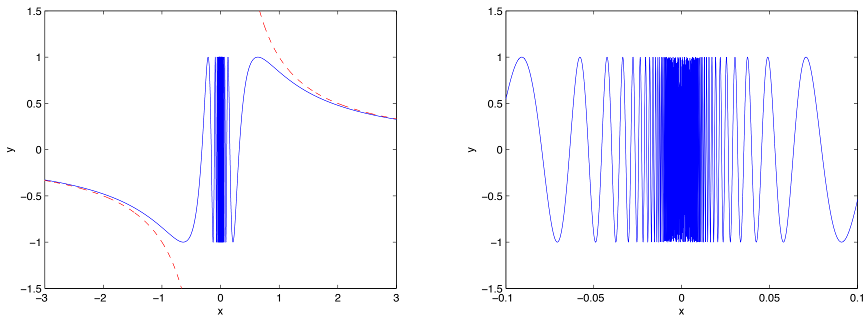

Figure 1. A plot of the function $y = \sin(1/x)$, with the hyperbola $y = 1/x$ shown in red, and a detail near the origin.

Example 6.8. Define the sign function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ by

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

Then the limit

$$\lim_{x \to 0} \operatorname{sgn} x$$

doesn't exist. To prove this, note that $(1/n)$ is a non-zero sequence such that $1/n \to 0$ and $\operatorname{sgn}(1/n) \to 1$ as $n \to \infty$, while $(-1/n)$ is a non-zero sequence such that $-1/n \to 0$ and $\operatorname{sgn}(-1/n) \to -1$ as $n \to \infty$. Since the sequences of sgn-values have different limits, Corollary 6.7 implies that the limit does not exist.

Example 6.9. The limit

$$\lim_{x \to 0} \frac{1}{x},$$

corresponding to the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$, doesn't exist. For example, if (x_n) is the non-zero sequence given by $x_n = 1/n$, then $1/n \to 0$ but the sequence of values (n) diverges to ∞.

Example 6.10. The limit

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right),$$

corresponding to the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \sin(1/x)$, doesn't exist. (See Figure 1.) For example, the non-zero sequences (x_n), (y_n) defined by

$$x_n = \frac{1}{2\pi n}, \qquad y_n = \frac{1}{2\pi n + \pi/2}$$

both converge to zero as $n \to \infty$, but the limits

$$\lim_{n \to \infty} f(x_n) = 0, \qquad \lim_{n \to \infty} f(y_n) = 1$$

are different.

Like sequences, functions must satisfy a boundedness condition if their limit is to exist. Before stating this condition, we define the supremum and infimum of a function, which are the supremum or infimum of its range.

Definition 6.11. If $f : A \to \mathbb{R}$ is a real-valued function, then

$$\sup_A f = \sup\{f(x) : x \in A\}, \qquad \inf_A f = \inf\{f(x) : x \in A\}.$$

A function is bounded if its range is bounded.

Definition 6.12. If $f : A \to \mathbb{R}$, then f is bounded from above if $\sup_A f$ is finite, bounded from below if $\inf_A f$ is finite, and bounded if both are finite. A function that is not bounded is said to be unbounded.

Example 6.13. If $f : [0,2] \to \mathbb{R}$ is defined by $f(x) = x^2$, then

$$\sup_{[0,2]} f = 4, \qquad \inf_{[0,2]} f = 0,$$

so f is bounded.

Example 6.14. If $f : (0,1] \to \mathbb{R}$ is defined by $f(x) = 1/x$, then

$$\sup_{(0,1]} f = \infty, \qquad \inf_{(0,1]} f = 1,$$

so f is bounded from below, not bounded from above, and unbounded. Note that if we extend f to a function $g : [0,1] \to \mathbb{R}$ by defining, for example,

$$g(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

then g is still unbounded on $[0,1]$.

Equivalently, a function $f : A \to \mathbb{R}$ is bounded if $\sup_A |f|$ is finite, meaning that there exists $M \geq 0$ such that

$$|f(x)| \leq M \text{ for every } x \in A.$$

If $B \subset A$, then we say that f is bounded from above on B if $\sup_B f$ is finite, with similar terminology for bounded from below on B, and bounded on B.

Example 6.15. The function $f : (0,1] \to \mathbb{R}$ defined by $f(x) = 1/x$ is unbounded, but it is bounded on every interval $[\delta, 1]$ with $0 < \delta < 1$. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is unbounded, but it is bounded on every finite interval $[a, b]$.

We also introduce a notion of being bounded near a point.

Definition 6.16. Suppose that $f : A \to \mathbb{R}$ and c is an accumulation point of A. Then f is locally bounded at c if there is a neighborhood U of c such that f is bounded on $A \cap U$.

Example 6.17. The function $f : (0,1] \to \mathbb{R}$ defined by $f(x) = 1/x$ is locally bounded at every $0 < c \leq 1$, but it is not locally bounded at 0.

Proposition 6.18. Suppose that $f : A \to \mathbb{R}$ and c is an accumulation point of A. If $\lim_{x \to c} f(x)$ exists, then f is locally bounded at c.

Proof. Let $\lim_{x\to c} f(x) = L$. Taking $\epsilon = 1$ in the definition of the limit, we get that there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < 1.$$

Let $U = (c - \delta, c + \delta)$. If $x \in A \cap U$ and $x \neq c$, then

$$|f(x)| \leq |f(x) - L| + |L| < 1 + |L|,$$

so f is bounded on $A \cap U$. (If $c \in A$, then $|f| \leq \max\{1 + |L|, |f(c)|\}$ on $A \cap U$.) □

As for sequences, boundedness is a necessary but not sufficient condition for the existence of a limit.

Example 6.19. The limit

$$\lim_{x\to 0} \frac{1}{x},$$

considered in Example 6.9 doesn't exist because the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$ is not locally bounded at 0.

Example 6.20. The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \sin\left(\frac{1}{x}\right)$$

is bounded, but $\lim_{x\to 0} f(x)$ doesn't exist.

6.2. Left, right, and infinite limits

We can define other kinds of limits in an obvious way. We list some of them here and give examples, whose proofs are left as an exercise. All these definitions can be combined in various ways and have obvious equivalent sequential characterizations.

Definition 6.21 (Right and left limits)**.** Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$. If $c \in \mathbb{R}$ is an accumulation point of $\{x \in A : x > c\}$, then f has the right limit

$$\lim_{x\to c^+} f(x) = L,$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$c < x < c + \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

If $c \in \mathbb{R}$ is an accumulation point of $\{x \in A : x < c\}$, then f has the left limit

$$\lim_{x\to c^-} f(x) = L,$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$c - \delta < x < c \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Equivalently, the right limit of f is the limit of the restriction $f|_{A^+}$ of f to the set $A^+ = \{x \in A : x > c\}$,

$$\lim_{x\to c^+} f(x) = \lim_{x\to c} f|_{A^+}(x),$$

and analogously for the left limit.

Example 6.22. For the sign function in Example 6.8, we have

$$\lim_{x\to 0^+} \operatorname{sgn} x = 1, \qquad \lim_{x\to 0^-} \operatorname{sgn} x = -1,$$

although the corresponding limit does not exist.

The existence and equality of the left and right limits implies the existence of the limit.

Proposition 6.23. Suppose that $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and $c \in \mathbb{R}$ is an accumulation point of both $\{x \in A : x > c\}$ and $\{x \in A : x < c\}$. Then

$$\lim_{x\to c} f(x) = L$$

if and only if

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L.$$

Proof. It follows immediately from the definitions that the existence of the limit implies the existence of the left and right limits with the same value. Conversely, if both left and right limits exists and are equal to L, then given $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$c - \delta_1 < x < c \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon,$$
$$c < x < c + \delta_2 \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Choosing $\delta = \min(\delta_1, \delta_2) > 0$, we get that

$$|x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon,$$

which show that the limit exists. □

Next we introduce some convenient definitions for various kinds of limits involving infinity. We emphasize that ∞ and $-\infty$ are not real numbers (what is $\sin\infty$, for example?) and all these definition have precise translations into statements that involve only real numbers.

Definition 6.24 (Limits as $x \to \pm\infty$)**.** Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$. If A is not bounded from above, then

$$\lim_{x\to\infty} f(x) = L$$

if for every $\epsilon > 0$ there exists an $M \in \mathbb{R}$ such that

$$x > M \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

If A is not bounded from below, then

$$\lim_{x\to-\infty} f(x) = L$$

if for every $\epsilon > 0$ there exists an $m \in \mathbb{R}$ such that

$$x < m \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Sometimes we write $+\infty$ instead of ∞ to indicate that it denotes arbitrarily large, positive values, while $-\infty$ denotes arbitrarily large, negative values.

It follows from the definitions that

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f\left(\frac{1}{t}\right), \qquad \lim_{x\to-\infty} f(x) = \lim_{t\to 0^-} f\left(\frac{1}{t}\right),$$

and it is often useful to convert one of these limits into the other.

Example 6.25. We have

$$\lim_{x\to\infty} \frac{x}{\sqrt{1+x^2}} = 1, \qquad \lim_{x\to-\infty} \frac{x}{\sqrt{1+x^2}} = -1.$$

Definition 6.26 (Divergence to $\pm\infty$)**.** Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

$$\lim_{x\to c} f(x) = \infty$$

if for every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that

$$0 < |x-c| < \delta \text{ and } x \in A \text{ implies that } f(x) > M,$$

and

$$\lim_{x\to c} f(x) = -\infty$$

if for every $m \in \mathbb{R}$ there exists a $\delta > 0$ such that

$$0 < |x-c| < \delta \text{ and } x \in A \text{ implies that } f(x) < m.$$

The notation $\lim_{x\to c} f(x) = \pm\infty$ is simply shorthand for the property stated in this definition; it does not mean that the limit exists, and we say that f diverges to $\pm\infty$.

Example 6.27. We have

$$\lim_{x\to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x\to\infty} \frac{1}{x^2} = 0.$$

Example 6.28. We have

$$\lim_{x\to 0^+} \frac{1}{x} = \infty, \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

How would you define these statements precisely? Note that

$$\lim_{x\to 0} \frac{1}{x} \neq \pm\infty,$$

since $1/x$ takes arbitrarily large positive (if $x > 0$) and negative (if $x < 0$) values in every two-sided neighborhood of 0.

Example 6.29. None of the limits

$$\lim_{x\to 0^+} \frac{1}{x}\sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0^-} \frac{1}{x}\sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0} \frac{1}{x}\sin\left(\frac{1}{x}\right)$$

is ∞ or $-\infty$, since $(1/x)\sin(1/x)$ oscillates between arbitrarily large positive and negative values in every one-sided or two-sided neighborhood of 0.

Example 6.30. We have

$$\lim_{x\to\infty}\left(\frac{1}{x}-x^3\right)=-\infty,\qquad \lim_{x\to-\infty}\left(\frac{1}{x}-x^3\right)=\infty.$$

How would you define these statements precisely and prove them?

6.3. Properties of limits

The properties of limits of functions follow from the corresponding properties of sequences and the sequential characterization of the limit in Theorem 6.6. We can also prove them directly from the ϵ-δ definition of the limit.

6.3.1. Order properties. As for limits of sequences, limits of functions preserve (non-strict) inequalities.

Theorem 6.31. Suppose that $f, g : A \to \mathbb{R}$ and c is an accumulation point of A. If

$$f(x) \le g(x) \qquad \text{for all } x \in A,$$

and $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

Proof. Let

$$\lim_{x\to c} f(x) = L, \qquad \lim_{x\to c} g(x) = M.$$

Suppose for contradiction that $L > M$, and let

$$\epsilon = \frac{1}{2}(L-M) > 0.$$

From the definition of the limit, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |f(x) - L| < \epsilon \qquad & \text{if } x \in A \text{ and } 0 < |x-c| < \delta_1, \\ |g(x) - M| < \epsilon \qquad & \text{if } x \in A \text{ and } 0 < |x-c| < \delta_2. \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$. Since c is an accumulation point of A, there exists $x \in A$ such that $0 < |x-a| < \delta$, and it follows that

$$\begin{aligned} f(x) - g(x) &= [f(x) - L] + L - M + [M - g(x)] \\ &> L - M - 2\epsilon \\ &> 0, \end{aligned}$$

which contradicts the assumption that $f(x) \le g(x)$. □

Finally, we state a useful "sandwich" or "squeeze" criterion for the existence of a limit.

Theorem 6.32. Suppose that $f, g, h : A \to \mathbb{R}$ and c is an accumulation point of A. If

$$f(x) \le g(x) \le h(x) \qquad \text{for all } x \in A$$

and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L,$$

then the limit of $g(x)$ as $x \to c$ exists and

$$\lim_{x\to c} g(x) = L.$$

We leave the proof as an exercise. We often use this result, without comment, in the following way: If

$$0 \le f(x) \le g(x) \quad \text{or} \quad |f(x)| \le g(x)$$

and $g(x) \to 0$ as $x \to c$, then $f(x) \to 0$ as $x \to c$.

It is essential for the bounding functions f, h in Theorem 6.32 to have the same limit.

Example 6.33. We have

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \qquad \text{for all } x \ne 0$$

and

$$\lim_{x\to 0}(-1) = -1, \qquad \lim_{x\to 0} 1 = 1,$$

but

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \qquad \text{does not exist.}$$

6.3.2. Algebraic properties.

Limits of functions respect algebraic operations.

Theorem 6.34. Suppose that $f, g : A \to \mathbb{R}$, c is an accumulation point of A, and the limits

$$\lim_{x\to c} f(x) = L, \qquad \lim_{x\to c} g(x) = M$$

exist. Then

$$\begin{aligned}
&\lim_{x\to c} kf(x) = kL && \text{for every } k \in \mathbb{R},\\
&\lim_{x\to c} [f(x) + g(x)] = L + M,\\
&\lim_{x\to c} [f(x)g(x)] = LM,\\
&\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M} && \text{if } M \ne 0.
\end{aligned}$$

Proof. We prove the results for sums and products from the definition of the limit, and leave the remaining proofs as an exercise. All of the results also follow from the corresponding results for sequences.

First, we consider the limit of $f + g$. Given $\epsilon > 0$, choose δ_1, δ_2 such that

$$\begin{aligned}
&0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon/2,\\
&0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |g(x) - M| < \epsilon/2,
\end{aligned}$$

and let $\delta = \min(\delta_1, \delta_2) > 0$. Then $0 < |x - c| < \delta$ implies that

$$|f(x) + g(x) - (L + M)| \le |f(x) - L| + |g(x) - M| < \epsilon,$$

which proves that $\lim(f + g) = \lim f + \lim g$.

To prove the result for the limit of the product, first note that from the local boundedness of functions with a limit (Proposition 6.18) there exists $\delta_0 > 0$ and

$K > 0$ such that $|g(x)| \leq K$ for all $x \in A$ with $0 < |x - c| < \delta_0$. Choose $\delta_1, \delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon/(2K),$$
$$0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |g(x) - M| < \epsilon/(2|L| + 1).$$

Let $\delta = \min(\delta_0, \delta_1, \delta_2) > 0$. Then for $0 < |x - c| < \delta$ and $x \in A$,

$$\begin{aligned} |f(x)g(x) - LM| &= |(f(x) - L)\, g(x) + L\,(g(x) - M)| \\ &\leq |f(x) - L|\ |g(x)| + |L|\, |g(x) - M| \\ &< \frac{\epsilon}{2K} \cdot K + |L| \cdot \frac{\epsilon}{2|L| + 1} \\ &< \epsilon, \end{aligned}$$

which proves that $\lim(fg) = \lim f \lim g$. □