Problems on limit of a function

Aug 25, 2019

Definition. A *deleted neighbourhood of* c is a set of the form $(c-\delta, c+\delta)\backslash\{c\}$ where $\delta > 0$. In some books it is denoted by $N'(c, \delta)$.

1. Suppose that f, g are functions such that $\lim_{x\to a} f(x) = \ell_1$ and $\lim_{x\to a} g(x) = \ell_2$. Show that $\lim_{x\to a}(f(x)+g(x)) = \ell_1+\ell_2$ and $\lim_{x\to a} f(x)g(x) = \ell_1\ell_2$. Furthermore, if $g(x) \neq 0$ in a deleted neighbourhood of a and $\ell_2 \neq 0$ then show that $\lim_{x\to a} f(x)/g(x) = \ell_1/\ell_2$.
2. Suppose that $f(x) \leq g(x)$ holds for every x in a deleted neighbourhood of a. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist then show that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.
3. Suppose that $\lim_{x\to a} f(x)$ exists (and is finite). Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in (a-\delta, a+\delta)\backslash\{a\}$ it holds that $|f(x) - f(y)| < \varepsilon$. (Note, here it is understood that x, y are in the domain of f, so that $f(x)$ and $f(y)$ are well-defined.)
4. Suppose that $\lim_{x\to a} f(x) = 0$ and $g(x)$ is bounded in a deleted neighbourhood of a. Show that $\lim_{x\to a} f(x)g(x) = 0$.
5. Suppose that $g(x) \leq f(x) \leq h(x)$ holds for every x in a deleted neighbourhood of a. If $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = \ell$ then show that $\lim_{x\to a} f(x) = \ell$.
6. Determine whether the following limits exist:

$$\lim_{x\to 0} x\cos\frac{1}{x}, \quad \lim_{x\to 0}\frac{\lfloor x\rfloor}{x}, \quad \lim_{x\to 0} x\left\lfloor\frac{1}{x}\right\rfloor, \quad \lim_{x\to\infty} x\left(\sqrt{x^2+1} - \sqrt[3]{x^3+1}\right).$$

7. Suppose that f is monotonic on an open neighbourhood around a. Show that the one sided limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to a-} f(x)$ must exist (need not be equal).
8. Suppose that f has IVP and f is monotone on an interval $[a, b]$. Show that f must be continuous on $[a, b]$.
9. Suppose that f is one-one and f has IVP on an interval $[a, b]$. Show that f must be continuous and monotone on $[a, b]$.

10. Let f, g be functions such that $f(g(x))$ is well-defined. If $\lim_{x\to a} g(x) = b$ and $\lim_{y\to b} f(x) = \ell$, is it necessary that $\lim_{x\to a} f(g(x)) = \ell$?

11. Let f, g be functions such that $f(g(x))$ is well-defined. If $\lim_{x\to a} g(x) = b$ and $\lim_{y\to b} f(x) = \ell$. We assume further that $g(x) \neq b$ for x in a deleted neighbourhood of a. Prove that $\lim_{x\to a} f(g(x)) = \ell$.

12. Suppose that f is a function which does not vanish in a deleted neighbourhood of a and satisfies $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} f(x)g(x) = \ell$. Show that

$$\lim_{x\to a}(1 + f(x))^{g(x)} = e^{\ell}.$$

13. Determine, with proof, the value of $\lim_{x\to\infty} \left(1 + \frac{1}{x^2 + \cos x}\right)^{x^2+x}$.

14. Suppose that $f : (-\delta, \delta) \to \mathbb{R}$ has the property that $\lim_{x\to 0}(f(x) + f(2x)) = 0$. Is it necessary that $\lim_{x\to 0} f(x) = 0$?

15. For any function $f : (-\delta, \delta) \to \mathbb{R}$ show that if $\lim_{x\to 0} f(x) = \ell$ then $\lim_{x\to 0} f(|x|) = \ell$. Is the converse true ?

16. Let $n \in \mathbb{N}$. Determine, with proof, the value of $\lim_{x\to 0} \frac{1 - \cos(x)\cos(2x)\cdots\cos(nx)}{x^2}$.

17. For $x \geq 1$ define $f(x) = \lfloor x \rfloor + (x - \lfloor x \rfloor)^{\lfloor x \rfloor}$. Show that f is continuous and strictly increasing. (Here $\lfloor x \rfloor$ denotes the largest integer $\leq x$.)

18. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a monotonic function such that $\lim_{x\to\infty} \frac{f(2x)}{f(x)} = 1$. Prove that for any $c \geq 1$, it holds that $\lim_{x\to\infty} \frac{f(cx)}{f(x)} = 1$.

19. Let $f : [0, +\infty) \to \mathbb{R}$ be a function such that for each $a \geq 0$, $\lim_{n\to\infty} f(a + n) = 0$. Is it necessary that $\lim_{n\to\infty} f(x) = 0$?

20. Suppose that $f : (-\delta, \delta) \to (0, \infty)$ has the property that $\lim_{x\to 0} \left(f(x) + \frac{1}{f(x)}\right) = 2$. Show that $\lim_{x\to 0} f(x) = 1$. (It is not given that this limit exists.)