Solutions to Class Test 2 (on Calculus)

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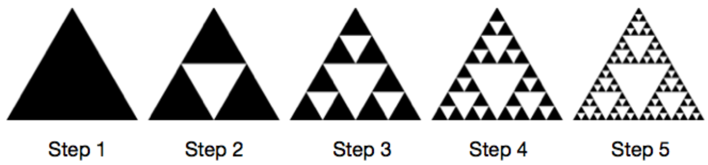
1. *Start with an equilateral triangle with unit side length. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle. Repeat last step with each of the remaining smaller triangles.*



Denote by $P(n)$ and $A(n)$ the perimeter and area of the existing portion of the triangle at the n-th step, e.g. $P(2) = 9/2$ unit and $A(2) = 3\sqrt{3}/16$ sq.unit. Find $\lim_{n\to\infty} P(n)$ and $\lim_{n\to\infty} A(n)$. Are you surprised?

Solution: It can be proved by induction on n that

$$P(n+1) = \frac{3}{2}P(n) \text{ and } A(n+1) = \frac{3}{4}A(n)$$

holds for every $n \geq 1$. Hence we get $P(n) = P(1)\left(\frac{3}{2}\right)^{n-1}$ and $A(n) = A(1)\left(\frac{3}{4}\right)^{n-1}$ for every $n \geq 1$. Since $0 < 3/4 < 1$, we know that $(3/4)^n \to 0$ as $n \to \infty$, which implies that $\lim_{n\to\infty} A(n) = 0$. On the other hand, since $3/2 > 1$, we know that $(3/2)^n$ diverges to $+\infty$ as $n \to \infty$. Therefore, we can write $\lim_{n\to\infty} P(n) = \infty$.

Comment: The result might be little surprising, because the limiting figure has an infinite perimeter but its area is zero. This limiting figure is known as the Sierpinski Triangle. To know more, you may search about it on Google.

2. *Let x_n be a sequence of integers such that $x_{k+1} \neq x_k$ holds for every $k \geq 1$. Show that x_n can not be a Cauchy sequence. Is it possible that x_n has a convergent subsequence?*

Solution: (Main idea: the sequence can't be Cauchy because for the terms to get arbitrarily closed, the sequence must be eventually constant which contradicts the given hypothesis.) Let, if possible, x_n be a Cauchy sequence. Then, for $\varepsilon = 1/2$, there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ holds for every $m, n \geq N$. But the terms of the sequence being integers, $|x_m - x_n| < 1/2$ forces $x_m = x_n$ for every $n \geq N$. This contradicts the fact that $x_{N+1} \neq x_N$.

The sequence may have a convergent subsequence. Consider, for example, the sequence $x_n = (-1)^n$ for $n \geq 1$.

3. *Suppose that* $f : [0, 2] \to \mathbb{R}$ *is continuous. Show that there exists* $a, b \in \mathbb{R}$ *such that* $b - a = 1$ *and* $f(b) - f(a) = \frac{1}{2}(f(2) - f(0))$.

   Solution: Consider the function $g(x) = f(x+1) - f(x)$. Observe that $g(0) + g(1) = f(2) - f(0)$. Therefore $\frac{1}{2}(f(2) - f(0))$ is just the average of $g(0)$ and $g(1)$; lets call it y. Since y lies between $g(0)$ and $g(1)$ and g is continuous, there exists $0 \leq a \leq 1$ such that $g(a) = y \implies f(a+1) - f(a) = y = \frac{1}{2}(f(2) - f(0))$.

4. Suppose that $f, g : [0, 1] \to [0, 1]$ are continuous functions such that $f(g(x)) = g(f(x))$ holds for every $x \in [0, 1]$.

   (a) (5 marks) Show that there exists $b \in [0, 1]$ such that $f(b) = b$.

   (b) (10 marks) Show that there exists $c \in [0, 1]$ such that $f(c) = g(c)$.

   Solution:

   (a) Consider $g(x) = f(x) - x$, $0 \leq x \leq 1$. Since $0 \leq f(x) \leq 1$ holds for every $0 \leq x \leq 1$, hence we have $g(0) \geq 0$ and $g(1) \leq 0$. Therefore, either one among $g(0)$ and $g(1)$ equals zero; else they have opposite signs which implies that there exists some $b \in (0, 1)$ such that $g(b) = 0$. And $g(b) = 0 \implies f(b) = b$.

   (b) We construct a sequence, starting with $a_0 = b$ (the same b as in part (a)) and define $a_{n+1} = g(a_n)$ for every $n \geq 1$. Observe that $f(a_0) = a_0$, $f(a_1) = f(g(a_0)) = g(f(a_0)) = g(a_0) = a_1$. We can induct on n to prove $f(a_n) = a_n$ for every $n \geq 1$. (Inductive step: Assuming $f(a_n) = a_n$, we get $f(a_{n+1}) = f(g(a_n)) = g(f(a_n)) = g(a_n) = a_{n+1}$.) Next we shall examine whether a_n is monotonic. Consider $h(x) = f(x) - g(x)$. If $h(0) = 0$ or $h(1) = 0$, or if $h(x)$ changes sign inside $[0, 1]$ then there exists $c \in [0, 1]$ such that $h(c) = 0$, as required. Let us assume to the contrary that $h(x)$ never vanishes inside $[0, 1]$. Since h is continuous, we must have either $h(x) > 0$ for all $x \in [0, 1]$; or $h(x) < 0$ for all $x \in [0, 1]$.

   Case 1: $h(x) > 0$, i.e. $f(x) > g(x)$ for all $x \in [a, b]$. In this case, we have $a_{n+1} = g(a_n) < f(a_n) = a_n$ for every $n \geq 0$. Therefore, a_n is monotonically decreasing in this case.

   Case 2: $h(x) < 0$, i.e. $f(x) < g(x)$ for all $x \in [a, b]$. In this case, we have $a_{n+1} = g(a_n) > f(a_n) = a_n$ for every $n \geq 0$. Therefore, a_n is monotonically increasing in this case.

Therefore, in both of the above cases, the sequence a_n is monotonic and it is bounded inside $[0, 1]$, hence $\lim a_n$ exists. Let us call $\lim a_n = a$. Letting $n \to \infty$ in $f(a_n) = a_n$ and $g(a_n) = a_{n+1}$, and using the fact that f, g are continuous, we get $f(a) = a = g(a)$. This is actually a contradiction, because we started with the assumption that $h(x) \neq 0$ on $[0, 1]$ and arrived at $h(a) = 0$. Hence our proof is complete.