2

Limit, Continuity and Differentiability of Functions

In this chapter we shall study limit and continuity of real valued functions defined on certain sets.

2.1 Limit of a Function

Suppose f is a real valued function defined on a subset D of $\mathbb{R}$. We are going to define *limit* of $f(x)$ as $x \in D$ *approaches* a point a which is not necessarily in D.

First we have to be clear about what we mean by the statement "$x \in D$ *approaches* a point a".

2.1.1 Limit point of a set $D \subseteq \mathbb{R}$

Definition 2.1 Let $D \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then a is said to be a **limit point** of D if for any $\delta > 0$, the interval $(a - \delta,\, a + \delta)$ contains atleast one point from D other than possibly a, i.e.,

$$D \cap \{x \in R : 0 < |x - a| < \delta\} \neq \varnothing.$$

□

Example 2.1 The statements in the following can be easily verified:

(i) Every point in an interval is its limit point.

(ii) If I is an open interval of finite length, then both the end points of I are limit points of I.

(iii) The set of all limit points of an interval I of finite length consists of points from I together with its endpoints.

(iv) If $D = \{x \in \mathbb{R} : 0 < |x| < 1\}$, then every point in the interval $[-1, 1]$ is a limit point of D.

(v) If $D = (0, 1) \cup \{2\}$, then 2 is not a limit point of D. The set of all limit points of D is the closed interval $[0, 1]$.

(vi) If $D = \{\frac{1}{n} : n \in \mathbb{N}\}$, then 0 is the only limit point of D.

(vii) If $D = \{n/(n+1) : n \in \mathbb{N}\}$, then 1 is the only limit point of D. □

For the later use, we introduce the following definition.

Definition 2.2 (i) For $a \in \mathbb{R}$, an open interval of the form $(a - \delta,\, a + \delta)$ for some $\delta > 0$ is called a **neighbourhood** of a; it is also called a **δ-neighbourhood** of a.

(ii) By a **deleted neighbourhood** of a point $a \in \mathbb{R}$ we mean a set of the form $D_\delta := \{x \in \mathbb{R} : 0 < |x - a| < \delta\}$ for some $\delta > 0$, i.e., the set $(a - \delta, a + \delta) \setminus \{a\}$. □

With the terminologies in the above definition, we can state the following:

- A point $a \in \mathbb{R}$ is a limit point of $D \subseteq \mathbb{R}$ if and only if every deleted neighbourhood of a contains at least one point of D.

  In particular, if D contains either a deleted neighbourhood of a or if D contains an open interval with one of its end points is a, then a is a limit point of D.

Now we give a characterization of limit points in terms of convergence of sequences.

Theorem 2.1 *A point $a \in \mathbb{R}$ is a limit point of $D \subseteq \mathbb{R}$ if and only if there exists a sequence (a_n) in $D \setminus \{a\}$ such that $a_n \to a$ as $n \to \infty$.*

Proof. Suppose $a \in \mathbb{R}$ is a limit point of D. Then for each $n \in \mathbb{N}$, there exists $a_n \in D \setminus \{a\}$ such that $a_n \in (a - 1/n, a + 1/n)$. Note that that $a_n \to a$.

Conversely, suppose that there exists a sequence (a_n) in $D\setminus\{a\}$ such that $a_n \to a$. Hence, for every $\delta > 0$, there exists $N \in \mathbb{N}$ such that $a_n \in (a-\delta, a+\delta)$ for all $n \geq N$. In particular, for $n \geq N$, $a_n \in (a - \delta, a + \delta) \cap (D \setminus \{a\})$. ▮

***Exercise* 2.1** Prove that a point $a \in \mathbb{R}$ is a limit point of $D \subset \mathbb{R}$ if and only if there exists a sequence (a_n) in D such that (a_n) is not eventually constant and $a_n \to a$ as $n \to \infty$. [Recall that a sequence (a_n) is said to be eventually constant if there exists $k \in \mathbb{N}$ such that $a_n = a_k$ for all $n \geq k$.] ◄

2.1.2 Limit of a function $f(x)$ as x approaches a

Definition 2.3 Let f be a real valued function defined on a set $D \subseteq \mathbb{R}$, and let $a \in \mathbb{R}$ be a limit point of D. We say that $b \in \mathbb{R}$ is a **limit of $f(x)$ as x approaches** a if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x \in D,\ 0 < |x - a| < \delta, \qquad (*)$$

and in that case we write

$$\lim_{x \to a} f(x) = b$$

or

$$f(x) \to b \quad \text{as} \quad x \to a.$$

□

The relations in (∗) in the above examples can also be written as

$$x \in D, \quad 0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - b| < \varepsilon.$$

***Exercise* 2.2** Thus, $\lim_{x \to a} f(x) = b$ if and only if for every open interval I_b containing b there exists an open interval I_a containing a such that

$$x \in I_a \cap (D \setminus \{a\}) \quad \Longrightarrow \quad f(x) \in I_b.$$

◀

CONVENTION: In the following, whenever we talk about limit of a function f as x approaches $a \in \mathbb{R}$, we assume that f is defined on a set $D \subseteq \mathbb{R}$ and a is a limit point of D.

Also, when we talk about $f(x)$, we assume that x belongs to the domain of f. For example, if we say that "$f(x)$ has certain property P for every x in an interval I", what we mean actualy is that "$f(x)$ has the property P for all $x \in I \cap D$, where D is he domain of f".

***Exercise* 2.3** Show that, a function cannot have more than one limits. ◀

Example 2.2 Let D be an interval and a is either in D or a is an end point of D.

(i) Let $f(x) = x$. Since

$$|f(x) - a| = |x - a| \quad \forall\, x \in D,$$

it follows that for any $\varepsilon > 0$, $|f(x) - a| < \varepsilon$ whenever $0 < |x - a| < \delta := \varepsilon$. Hence, $\lim_{x \to a} f(x) = a$.

(ii) Let $f(x) = x^2$ and $\varepsilon > 0$ be given. We show that $\lim_{x \to a} f(x) = a^2$. Note that

$$|f(x) - a^2| = (|x| + |a|)|x - a| \quad \forall\, x \in D,\, x \neq a.$$

Since $|x| \leq |x - a| + |a| \leq 1 + |a|$ whenever $|x - a| < 1$, we have

$$|f(x) - a^2| = (1 + 2|a|)|x - a| \quad \forall\, x \in D,\, 0 < |x - a| \leq 1.$$

Therefore,

$$x \in D,\, 0 < |x - a| \leq 1,\, (1 + 2|a|)|x - a| < \varepsilon \quad \Longrightarrow \quad |f(x) - a^2| < \varepsilon.$$

Thus,

$$x \in D,\, 0 < |x - a| < \delta := \min\{1,\, \varepsilon/(1 + 2|a|)\} \quad \Longrightarrow \quad |f(x) - a^2| < \varepsilon.$$

Hence, $\lim_{x \to a} f(x) = a^2$. □

More examples will be considered in Section 2.1.4 after proving some properties of the limit. Before that let us ask the following question.

Question: Suppose f is a real valued function defined on an interval I and $a \in I$. What do we mean by the statement that "$\lim_{x\to a} f(x)$ does not exist"?

It means the following: For any $b \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for any $\delta > 0$, there is atleast one $x_\delta \in (a-\delta, a+\delta)$ such that $f(x_\delta) \notin (b-\varepsilon, b+\varepsilon)$.

We illustrate this by a simple example.

Example 2.3 Let $f : [-1,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0, & -1 \le x \le 0, \\ 1, & 0 < x \le 1. \end{cases}$ We show that $\lim_{x\to 0} f(x)$ does not exist. For this let $b \in \mathbb{R}$. Let us consider the following cases:

Case (i): $b = 0$. In this case, if $0 < \varepsilon < 1$, then $(b-\varepsilon, b+\varepsilon)$ does not contain 1 so that $f(x) \notin (b-\varepsilon, b+\varepsilon)$ for any $x > 0$.

Case (ii): $b = 1$. In this case, if $0 < \varepsilon < 1$, then $(b-\varepsilon, b+\varepsilon)$ does not contain 0 so that $f(x) \notin (b-\varepsilon, b+\varepsilon)$ for any $x < 0$.

Case (iii): $b \ne 0$, $b \ne 1$. In this case, if $0 < \varepsilon < \min\{|b|, |b-1|\}$, then $(b-\varepsilon, b+\varepsilon)$ does not contain 0 and 1 so that $f(x) \notin (b-\varepsilon, b+\varepsilon)$ for any $x \ne 0$.

Thus, b is not a limit of $f(x)$ as x approaches 0. □

Before going further, let us observe a property which would be used in the due course.

Theorem 2.2 *If $\lim_{x\to a} f(x) = b$, then there exists a deleted neighbourhood D_δ of a and $M > 0$ such that $|f(x)| \le M$ for all $x \in D_\delta \cap D$.*

Proof. Suppose $\lim_{x\to a} f(x) = b$. Then there exists a deleted neighbourhood D_δ of a such that $|f(x) - b| < 1$ for all $x \in D \cap D_\delta$. Hence,

$$|f(x)| \le |f(x) - b| + |b| < 1 + |b| \quad \forall\, x \in D \cap D_\delta.$$

Thus, $|f(x)| \le M = 1 + |b|$ for all $x \in D_\delta \cap D$. ∎

2.1.3 Limit of a function in terms of sequences

Let a be a limit point of $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$.

Suppose $\lim_{x\to a} f(x) = b$. Since a is a limit point of D, we know that there exists a sequence (x_n) in $D \setminus \{a\}$ such that $x_n \to a$. Does $f(x_n) \to b$? The answer is "yes". In fact, we have more!

Theorem 2.3 *If $\lim_{x\to a} f(x) = b$, then for every sequence (x_n) in D such that $x_n \to a$, we have $f(x_n) \to b$.*

Proof. Suppose $\lim_{x\to a} f(x) = b$. Let (x_n) be a sequence in D such that $x_n \to a$. Let $\varepsilon > 0$ be given. We have to show that there exists $n_0 \in \mathbb{N}$ such that $|f(x_n) - b| < \varepsilon$ for all $n \geq n_0$.

Since $\lim_{x\to a} f(x) = b$, we know that there exists $\delta > 0$ such that

$$x \in D,\ 0 < |x - a| < \delta \Longrightarrow |f(x) - b| < \varepsilon. \tag{$*$}$$

Also, since $x_n \to a$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n \geq n_0$. Hence, from $(*)$, we have $|f(x_n) - b| < \varepsilon$ for all $n \geq n_0$. ∎

The converse of the above theorem is also true.

Theorem 2.4 *If for every sequence (x_n) in D which converges to a, the sequence $(f(x_n))$ converges to b, then $\lim_{x\to a} f(x) = b$.*

Proof. Suppose for every sequence (x_n) in D which converges to a, the sequence $(f(x_n))$ converges to b. Assume for a moment that f does not have the limit b as x approaches a. Then, by the definition of the limit, there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exists at least one $x_\delta \in D$ such that

$$0 < |x_\delta - a| < \delta \quad \text{and} \quad |f(x_\delta) - b| > \varepsilon_0.$$

In particular, for every $n \in \mathbb{N}$, there exists $x_n \in D$ such that

$$0 < |x_n - a| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - b| > \varepsilon_0.$$

Thus, $x_n \to a$ but $f(x_n) \not\to b$. This is a contradiction to our hypothesis. ∎

Remark 2.1 Here are some implications of the first part of Theorem 2.3. Suppose (x_n) is a a sequence in $D \setminus \{a\}$ such that $x_n \to a$.

1. If $(f(x_n))$ does not converge, then $\lim_{x\to a} f(x)$ does not exist.
2. If $(f(x_n))$ does not converge to a given $b \in \mathbb{R}$, then either $\lim_{x\to a} f(x)$ does not exist or $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} f(x) \neq b$.
3. If (y_n) is another sequence in $D \setminus \{a\}$ which converges to a and the sequences $(f(x_n))$ and $(f(y_n))$ converge to different points, then $\lim_{x\to a} f(x)$ does not exist.

If we are able to show the convergence of $(f(x_n))$ to some b for any arbitrary (*not for a specific*) sequence (x_n) in $D \setminus \{a\}$ which converges to a, then by second part of Theorem 2.3, we can assert that $\lim_{x\to a} f(x) = b$. ♦

Example 2.4 Consider the function f in Example 2.3, i.e., $f : [-1, 1] \to \mathbb{R}$ is defined by $f(x) = \begin{cases} 0, & -1 \le x \le 0, \\ 1, & 0 < x \le 1. \end{cases}$

Suppose (x_n) is a sequence of negative numbers and (y_n) is a sequence of positive numbers such that both of them converge to 0. Then we have $f(x_n) = 0$ and $f(y_n) = 1$ for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$ exist, but they are different. Hence $\lim_{x\to 0} f(x)$ does not exist. □

2.1.4 Some properties

The following two theorems can be proved using Theorems 2.3 and 2.4, and the results on convergence of sequences of real numbers.

Theorem 2.5 *We have the following.*

(i) *If* $\lim_{x\to a} f(x) = b$ *and* $\lim_{x\to a} g(x) = c$, *then*

$$\lim_{x\to a}[f(x) + g(x)] = b + c, \qquad \lim_{x\to a} f(x)g(x) = bc.$$

(ii) *If* $\lim_{x\to a} f(x) = b$ *and* $b \ne 0$, *then* $f(x) \ne 0$ *in a deleted neighbourhood of* a *and*

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{b}.$$

Theorem 2.6 (Sandwich theorem) *If* f *and* g *have the same limit* b *as* x *approaches* a, *and if* h *is a function such that* $f(x) \le h(x) \le g(x)$ *for all* x *in a deleted neighbourhood of* a, *then* $\lim_{x\to a} h(x) = b$.

The following two corollaries are immediate from Theorem 2.5.

Corollary 2.7 *If* $\lim_{x\to a} f(x) = b$, $\lim_{x\to a} g(x) = c$, *and* $c \ne 0$, *then* g *is nonzero in a deleted neighbourhood of* c *and*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{b}{c}.$$

Corollary 2.8 *If* $\lim_{x\to a} f(x) = b$, $\lim_{x\to a} g(x) = c$ *and* $f(x) \ge g(x)$ *for all* x *in a deleted neighbourhood of* a, *then* $b \ge c$.

***Exercise* 2.4** Write detailed proof of Theorem 2.5, Theorem 2.6 and Corollary 2.7 and Corollary 2.8. ◀

Theorem 2.9 *Suppose* $\lim_{x\to a} f(x) = b$ *and* $\lim_{y\to b} g(y) = c$. *If* D_1 *and* D_2 *are the domains of* f *and* g *respectively, and if* $f(x) \in D_2 \setminus \{b\}$ *for every* $x \in D_1 \setminus \{a\}$, *then* $\lim_{x\to a} g(f(x)) = c$.

Proof. By Theorem 2.6, it is enough to prove that for any sequence (x_n) in $D_1 \setminus \{a\}$ such that $x_n \to a$, we have $g(f(x_n)) \to c$. So, let (x_n) be in $D_1 \setminus \{0\}$ such that $x_n \to a$. Since $\lim_{x\to a} f(x) = b$, by Theorem 2.5, $f(x_n) \to b$. Let $y_n = f(x_n)$, $n \in \mathbb{N}$. By the assumption, $y_n \in D_2 \setminus \{b\}$ for all $n \in \mathbb{N}$. Since $\lim_{y\to b} g(y) = c$ and $y_n \to b$, again by Theorem 2.5, $g(y_n) \to c$. Thus we obtained $g(f(x_n) \to c$, which completes the proof. ∎

Alternate proof using $\varepsilon - \delta$ arguments. Let $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \Longrightarrow |g(y) - c| < \varepsilon.$$

Also, let $\delta_2 > 0$ be such that

$$0 < |x - a| < \delta_2 \Longrightarrow |f(x) - b| < \delta_1.$$

Hence, along with the given condition that $f(x) \in D_2 \setminus \{b\}$ for every $x \in D_1 \setminus \{a\}$,

$$0 < |x - a| < \delta_2 \Longrightarrow 0 < |f(x) - b| < \delta_1 \Longrightarrow |g(f(x)) - c| < \varepsilon.$$

This completes the proof. ∎

***Exercise* 2.5** Suppose φ is a function defined in a neighbourhood of a point x_0 such that $\lim_{x\to x_0} \varphi(x) = x_0$. If f is also a function defined in a neighbourhood of x_0 and $\lim_{x\to x_0} f(x)$ exists, then prove that $\lim_{x\to x_0} f(\varphi(x))$ exists and

$$\lim_{x\to x_0} f(\varphi(x)) = \lim_{x\to x_0} f(x).$$

◀

Example 2.5 If $f(x)$ is a polynomial, say $f(x) = a_0 + a_1 x + \ldots + a_k x^k$, then for any $a \in \mathbb{R}$,

$$\lim_{x\to a} f(x) = f(a).$$

We obtain this by using Theorem 2.5. Let us show the same by using the definition, i.e., using $\varepsilon - \delta$ arguments: Let $b = f(a)$ and let $\varepsilon > 0$ be given. We have to find $\delta > 0$ such that $|x - a| < \delta \Longrightarrow |f(x) - b| < \varepsilon$. Note that

$$f(x) - f(a) = a_1(x - a) + a_2(x^2 - a^2) + \ldots + a_k(x^k - a^k),$$

where

$$x^n - a^n = (x - a)[x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1}].$$

Now, suppose $|x - a| < 1$. Then we have $|x| < 1 + |a|$ so that

$$|x^{n-j}a^{j-1}| < (1 + |a|)^{n-1}$$

and hence,

$$|x^n - a^n| < |x - a| n (1 + |a|)^{n-1}.$$

Thus, $|x-a|<1$ implies

$$|f(x)-f(a)| \le |x-a|\Big(|a_1|+|a_2|2(1+|a|)+\ldots+|a_k|k(1+|a|)^{k-1}\Big),$$

Therefore, taking $\alpha := |a_1|+|a_2|2(1+|a|)+\ldots+|a_k|k(1+|a|)^{k-1}$, we have

$$|f(x)-f(a)|<\varepsilon \quad \text{whenever} \quad |x-a|<\delta := \min\{1, \varepsilon/\alpha\}.$$

□

Example 2.6 Let $D=\mathbb{R}\setminus\{2\}$ and $f(x)=\frac{x^2-4}{x-2}$. Then $\lim\limits_{x\to 2} f(x)=4$.

Note that, for $x\neq 2$,

$$f(x)=\frac{(x+2)(x-2)}{x-2}=(x+2).$$

Hence, for $\varepsilon>0$, $|f(x)-4|<\varepsilon$ whenever $|x-2|<\delta:=\varepsilon$. □

Example 2.7 Let $D=\mathbb{R}\setminus\{0\}$ and $f(x)=\frac{1}{x}$. Then $\lim\limits_{x\to 0} f(x)$ does not exist. To see this consider the sequence (x_n) with $x_n=1/n$ for $n\in\mathbb{N}$. Then we have $x_n\to 0$ but $\{f(x_n)\}$ diverges to infinity. Therefore, by Theorem 2.3, $\lim\limits_{x\to 0} f(x)$ does not exist.

Alternatively, for any $b\in\mathbb{R}$,

$$|f(x)-b| \ge |f(x)|-|b|>1 \quad \text{whenever} \quad |f(x)|>1+|b|.$$

But,

$$|f(x)|>1+|b| \iff |x|<\frac{1}{1+|b|}.$$

Thus, for any $b\in\mathbb{R}$,

$$|f(x)-b|>1 \quad \text{whenever} \quad |x|<\frac{1}{1+|b|}.$$

Thus, we have proved that it is not possible to find a $\delta>0$ such that $|f(x)-b|<1$ for all x with $|x|<\delta$. □

Example 2.8 We show that (i) $\lim\limits_{x\to 0}\sin(x)=0$ and (ii) $\lim\limits_{x\to 0}\cos(x)=1$.

From the graph of the function $\sin x$, it is clear that

$$-\frac{\pi}{2}<x<0 \Longrightarrow 0<|\sin x|<|x|.$$

Hence, from Theorem 2.6, we have $\lim\limits_{x\to 0}|\sin x|=0$. Thus, $\lim\limits_{x\to 0}\sin(x)=0$.

Also, since $\cos x = 1-2\sin^2(x/2)$ and $\lim\limits_{x\to 0}\sin(x/2)=0$, Theorem 2.5(i) implies $\lim\limits_{x\to 0}\cos x=1$. □

Example 2.9 We show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

It can be seen, using the graph of $\sin x$ that

$$0 < x < \frac{\pi}{2} \Longrightarrow \sin x < x < \tan x.$$

Hence,

$$0 < x < \frac{\pi}{2} \Longrightarrow \cos x < \frac{\sin x}{x} < 1.$$

Since $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$ and $\cos(-x) = \cos x$, it follows that

$$0 < |x| < \frac{\pi}{2} \Longrightarrow \cos x < \frac{\sin x}{x} < 1.$$

Therefore, by Theorem 2.5(iv) and Example 2.8(ii), we have $\lim_{x\to 0} \frac{\sin x}{x} = 1$. □

Remark 2.2 In the above two examples we have used some properties of the functions $\sin x$, $\cos x$ and $\tan x$, though we have not defined these functions formally. We shall define these functions formally in the due course. ♦

***Exercise* 2.6** Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x)+f(y)$. Suppose $\lim_{x\to 0} f(x)$ exists. Prove that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to c} f(x) = f(c)$ for every $c \in \mathbb{R}$.

Hint: Use the facts that $f(2x) = 2f(x)$, Theorem 2.9 and $f(x) - f(c) = f(x - c)$. ◄

***Exercise* 2.7** Suppose φ is a function defined in a neighbourhood I_0 of a point x_0 such that

$$x \in I_0,\ |x - x_0| < r \Longrightarrow |\varphi(x) - x_0| < r \qquad \forall\, r > 0.$$

If f is also a function defined in a neighbourhood of x_0 and $\lim_{x\to x_0} f(x)$ exists, then prove that $\lim_{x\to x_0} f(\varphi(x))$ exists and $\lim_{x\to x_0} f(\varphi(x)) = \lim_{x\to x_0} f(x)$. ◄

2.1.5 Left limit and right limit

Definition 2.4 Let f be a real valued function defined on a set $D \subseteq \mathbb{R}$, and let $a \in \mathbb{R}$ be a limit point of D.

(i) We say that $f(x)$ **has the left limit** $b \in \mathbb{R}$ **as** x **approaches** a if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x \in D,\ a - \delta < x < a,$$

and in that case we write $\lim_{x\to a-} f(x) = b$.

(ii) We say that $f(x)$ **has the right limit** $b \in \mathbb{R}$ **as** x **approaches** a if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x \in D,\ a < x < a + \delta,$$

and in that case we write $\lim_{x\to a+} f(x) = b$. □

We shall use the notations:

$$f(x_0-) := \lim_{x\to x_0-} f(x), \qquad f(x_0-) := \lim_{x\to x_0+} f(x)$$

whenever the above limits exists.

We have the following characterizations in terms of sequences (*Verify*):

1. $\lim\limits_{x\to a-} f(x) = b$ if and only if for every sequence (x_n) in $D\setminus\{a\}$,
$$x_n < a \quad \forall n\in\mathbb{N}, \quad x_n \to a \quad\Longrightarrow\quad f(x_n)\to b.$$

2. $\lim\limits_{x\to a+} f(x) = b$ if and only if for every sequence (x_n) in $D\setminus\{a\}$,
$$x_n > a \quad \forall n\in\mathbb{N}, \quad x_n \to a \quad\Longrightarrow\quad f(x_n)\to b.$$

The proof of the following theorem is left as an exercise.

Theorem 2.10 *Let f be a real valued function defined on a set $D\subseteq\mathbb{R}$, and let $a\in\mathbb{R}$ be a limit point of D. Then $\lim\limits_{x\to a} f(x)$ exists if and only if $\lim\limits_{x\to a-} f(x)$ and $\lim\limits_{x\to a+} f(x)$ exist and $\lim\limits_{x\to a-} f(x) = \lim\limits_{x\to a+} f(x)$, and in that case*

$$\lim_{x\to a} f(x) = \lim_{x\to a-} f(x) = \lim_{x\to a+} f(x).$$

In view of the above theorem, if $\lim\limits_{x\to a-} f(x)$ does not exist or $\lim\limits_{x\to a+} f(x)$ does not exist or both $\lim\limits_{x\to a-} f(x)$ and $\lim\limits_{x\to a+} f(x)$ exist but $\lim\limits_{x\to a-} f(x) \neq \lim\limits_{x\to a+} f(x)$, then $\lim\limits_{x\to a} f(x)$ does not exist.

Example 2.10 Let us consider the a few examples to illustrate Theorem 2.10.

(i) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/x, & x>0, \\ 1, & x\le 0. \end{cases}$$

In this case we see that $\lim\limits_{x\to 0^-} f(x) = 1$, but $\lim\limits_{x\to 0^+} f(x)$ does not exist.

(ii) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/x, & x<0, \\ 1, & x\ge 0. \end{cases}$$

In this case we see that $\lim\limits_{x\to 0^+} f(x) = 1$, but $\lim\limits_{x\to 0^-} f(x)$ does not exist.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/x, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

In this case, both $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ do not exist.

(iv) Let f be as in Example 2.3, that is, $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & -1 \leq x \leq 0, \\ 1, & 0 < x \leq 1. \end{cases}$$

In this case both $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ exist, but $\lim_{x \to 0} f(x)$ does not exist. □

2.1.6 Limit at ∞ and at $-\infty$

Definition 2.5 Suppose a function f is defined on an interval of the form (a, ∞) for some $a \in \mathbb{R}$. Then we say that $f(x)$ **has the limit** b **as** $x \to \infty$, if for every $\varepsilon > 0$, there exits $M > a$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x > M,$$

and in that case we write $\lim_{x \to \infty} f(x) = b$ □

Definition 2.6 Suppose a function f is defined on an interval of the form $(-\infty, a)$ for some $a \in \mathbb{R}$. Then we say that $f(x)$ **has the limit** b **as** $x \to -\infty$, if for every $\varepsilon > 0$, there exits $M < a$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x < M,$$

and in that case we $\lim_{x \to -\infty} f(x) = b$, □

Definition 2.7 For $a \in \mathbb{R}$, the interval (a, ∞) is called a **neighbourhood of** ∞ and the interval $(-\infty, a)$ is called a **neighbourhood of** $-\infty$. □

Now, we give the sequential characterization of limits at ∞ and at $-\infty$.

Theorem 2.11 *The following hold.*

(i) *Let f is defined in a neighbourhood D_1 of ∞ and $b \in \mathbb{R}$. Then $\lim_{x \to \infty} f(x) = b$ if and only if for every sequence (x_n) in D_1 with $x_n \to \infty$, $f(x_n) \to b$.*

(ii) *Let f is defined in a neighbourhood D_2 of $-\infty$ and $b \in \mathbb{R}$. Then $\lim_{x \to -\infty} f(x) = b$ if and only if for every sequence (x_n) in D_2 with $x_n \to \infty$, $f(x_n) \to b$.*

Proof. Suppose $\lim_{x \to \infty} f(x) = b$, and let (x_n) be in D_1 such that $x_n \to \infty$. Let $\varepsilon > 0$ be given. To show show that there exists $N \in \mathbb{N}$ such that $|f(x_n) - b| < \varepsilon$ for all $n \geq N$. Since $\lim_{x \to \infty} f(x) = b$, there exists $M > 0$ such that

$$x \in D_1, \, x > M \quad \Longrightarrow \quad |f(x) - b| < \varepsilon. \tag{1}$$

Since $x_n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \Longrightarrow \quad x_n > M. \tag{2}$$

From (1) and (2) above we have

$$n \geq n_0 \quad \Longrightarrow \quad |f(x_n) - b| < \varepsilon.$$

Thus, we have proved (i). Analogously we obtain proof of (ii). ∎

The following can be verified by applying Theorem 2.11.

1. If $\lim_{x\to\infty} f(x) = b$ and $\lim_{x\to\infty} g(x) = c$, then

$$\lim_{x\to\infty} [f(x) + g(x)] = b + c, \quad \lim_{x\to\infty} f(x)g(x) = bc.$$

2. If If $\lim_{x\to\infty} f(x) = b$, $\lim_{x\to\infty} g(x) = c$ and $c \neq 0$, then there exists $M_0 > 0$ such that $g(x) \neq 0$ for all $x > M_0$ and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{b}{c}.$$

Example 2.11 (i) We show that $\lim_{x\to\infty} \frac{1}{x} = 0$. Taking $f(x) = \frac{1}{x}$ for $x \neq 0$, $b = 0$ and $\varepsilon > 0$, we observe that

$$|f(x) - b| < \varepsilon \iff \frac{1}{|x|} < \varepsilon \iff |x| > \frac{1}{\varepsilon}.$$

Hence,

$$x > 1/\varepsilon \quad \Longrightarrow \quad |x| > 1/\varepsilon \quad \Longrightarrow \quad |f(x) - b| < \varepsilon.$$

This shows that $|f(x) - b| < \varepsilon$ whenever $x > M := 1/\varepsilon$.

(ii) We show that $\lim_{x\to-\infty} \frac{1}{x} = 0$. As before, taking $f(x) = \frac{1}{x}$ for $x \neq 0$, $b = 0$ and $\varepsilon > 0$, we observe that

$$|f(x) - b| < \varepsilon \iff \frac{1}{|x|} < \varepsilon \iff |x| > \frac{1}{\varepsilon}.$$

Hence,

$$x < -1/\varepsilon \quad \Longrightarrow \quad |x| > 1/\varepsilon \quad \Longrightarrow \quad |f(x) - b| < \varepsilon.$$

This shows that $|f(x) - b| < \varepsilon$ whenever $x < M := -1/\varepsilon$.

(iii) We show that $\lim_{x\to\infty} \frac{1}{x^2} = 0$. Taking $f(x) = \frac{1}{x^2}$ for $x \neq 0$, $b = 0$ and $\varepsilon > 0$, we observe that

$$|f(x) - b| < \varepsilon \iff \frac{1}{x^2} < \varepsilon \iff |x| > \frac{1}{\sqrt{\varepsilon}}.$$

Hence,

$$x > 1/\sqrt{\varepsilon} \quad \Longrightarrow \quad |x| > 1/\sqrt{\varepsilon} \quad \Longrightarrow \quad |f(x) - b| < \varepsilon.$$

This shows that $|f(x) - b| < \varepsilon$ whenever $x > M := 1/\sqrt{\varepsilon}$.

(iv) We show that $\lim\limits_{x\to\infty} \dfrac{1+x}{1+x^2} = 0$. Let $f(x) = \dfrac{1+x}{1+x^2}$ for $x \in \mathbb{R}$. The, by (i) and (iii) above,

$$f(x) = \frac{1+x}{1+x^2} = \frac{1/x^2 + 1/x}{1/x^2 + 1} \to \frac{0}{1} = 0.$$

(v) We show that $\lim\limits_{x\to\infty} \dfrac{1+x}{1-x} = -1$. Let $f(x) = \dfrac{1+x}{1-x}$ for $x \neq 1$. By (i) above,

$$f(x) = \frac{1+x}{1-x} = \frac{1/x + 1}{1/x - 1} \to \frac{1}{-1} = -1.$$

(vi) We show that $\lim\limits_{x\to\infty} \dfrac{1+2x}{1+3x} = \dfrac{2}{3}$. Let $f(x) = \dfrac{1+2x}{1+3x}$ for $x \neq -1/3$. Then, by (i),

$$f(x) = \frac{1+2x}{1+3x} = \frac{1/x + 2}{1/x + 3} = \frac{2}{3}.$$

□

Definition 2.8 We define the following:

1. $\lim\limits_{x\to a} f(x) = \infty$ if for every $M > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow f(x) > M.$$

2. $\lim\limits_{x\to a} f(x) = -\infty$ if for every $M > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow f(x) < -M.$$

3. $\lim\limits_{x\to+\infty} f(x) = \infty$ if for every $M > 0$, there exists $\alpha > 0$ such that

$$x > \alpha \Longrightarrow f(x) > M.$$

4. $\lim\limits_{x\to+\infty} f(x) = -\infty$ if for every $M > 0$, there exists $\alpha > 0$ such that

$$x > \alpha \Longrightarrow f(x) < -M.$$

5. $\lim\limits_{x\to-\infty} f(x) = \infty$ if for every $M > 0$, there exists $\alpha > 0$ such that

$$x < -\alpha \Longrightarrow f(x) > M.$$

6. $\lim_{x\to-\infty} f(x) = -\infty$ if for every $M > 0$, there exists $\alpha > 0$ such that

$$x < -\alpha \Longrightarrow f(x) < -M.$$

□

It can be easily shown (*Verify*) that

$$\lim_{x\to a} f(x) = \infty \iff \lim_{x\to a}[-f(x)] = -\infty,$$
$$\lim_{x\to+\infty} f(x) = \infty \iff \lim_{x\to+\infty}[-f(x)] = -\infty,$$
$$\lim_{x\to-\infty} f(x) = \infty \iff \lim_{x\to-\infty}[-f(x)] = -\infty.$$

Example 2.12 (i) We show that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Taking $f(x) = \frac{1}{x^2}$ for $x \neq 0$ and $M > 0$, we observe that

$$f(x) > M \iff \frac{1}{x^2} > M \iff |x| < \frac{1}{\sqrt{M}}.$$

Hence, for $0 < \delta < 1/\sqrt{M}$,

$$|x| < \delta \quad\Longrightarrow\quad |x| < \frac{1}{\sqrt{M}} \quad\Longrightarrow\quad f(x) > M.$$

Thus, $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

(ii) We show that $\lim_{x\to 1} \left|\frac{1+x}{1-x}\right| = \infty$.

Let $f(x) = \left|\frac{1+x}{1-x}\right|$ for $x \neq 1$. Then for $M > 0$,

$$f(x) = \left|\frac{1+x}{1-x}\right| > M \iff |1-x| < \frac{|1+x|}{M}$$

and

$$|1+x| = |2-(1-x)| \geq 2 - |1-x| > 1 \quad \text{whenever} \quad |x-1| < 1.$$

Hence

$$|x-1| < 1 \quad \text{and} \quad |x-1| < \frac{1}{M} \quad\Longrightarrow\quad |1-x| < \frac{|1+x|}{M} \quad\Longrightarrow\quad f(x) > M$$

Thus,

$$|x-1| < \delta := \min\{1, 1/M\} \quad\Longrightarrow\quad f(x) > M$$

showing that $\lim_{x\to 1} \left|\frac{1+x}{1-x}\right| = \infty$.

(iii) Let $f(x) = x^2$, $x \in \mathbb{R}$. We show that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$.

For $M > 0$,

$$f(x) = x^2 > M \iff |x| > \sqrt{M}.$$

Thus,

$$x > \sqrt{M} \quad \Longrightarrow f(x) > M$$

and

$$x < -\sqrt{M} \quad \Longrightarrow f(x) > M.$$

□

Example 2.13 Recall that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists, and we denoted it by e. Now we show that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

So, let $\varepsilon > 0$ be given. We have to find an $M > 0 \in \mathbb{N}$ such that

$$e - \varepsilon < \left(1+\frac{1}{x}\right)^x < e + \varepsilon \quad \text{whenever} \quad x > M. \tag{$*$}$$

Now, we can see that, for every $n \in \mathbb{N}$, if $x \in \mathbb{R}$ is such that $n \le x \le n+1$, then

$$1 + \frac{1}{n+1} \le 1 + \frac{1}{x} \le 1 + \frac{1}{n}$$

so that

$$\left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1}.$$

Thus is is same as

$$\alpha_n \le \left(1+\frac{1}{x}\right)^x \le \beta_n,$$

where

$$\alpha_n := \left(1+\frac{1}{n+1}\right)^{-1}\left(1+\frac{1}{n+1}\right)^{n+1}, \qquad \beta_n := \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right).$$

We know that $\alpha_n \to e$ and $\beta_n \to e$ as $n \to \infty$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$e - \varepsilon < \alpha_n < e + \varepsilon, \qquad e - \varepsilon < \beta_n < e + \varepsilon$$

for all $n \ge n_0$. Now, for $x > n_0$, let $n \ge n_0$ be such that $n \le x \le n+1$. Then we have

$$e - \varepsilon < \alpha_n \le \left(1+\frac{1}{x}\right)^x \le \beta_n < e + \varepsilon.$$

Thus, we obtained an $M := n_0 > 0$ such that

$$e - \varepsilon < \left(1+\frac{1}{x}\right)^x < e + \varepsilon \quad \text{whenever} \quad x > M.$$

Thus, we have proved $(*)$. □

***Exercise* 2.8** Suppose (α_n) and (β_n) are sequences of positive real numbers and f is a (real valued) function defined on $(0, \infty)$ having the following property: For $n \in \mathbb{N}$, $x \in \mathbb{R}$,

$$n < x < n+1 \implies \alpha_n \leq f(x) \leq \beta_n.$$

If (α_n) and (β_n) converge to the same limit, say b, then $\lim_{x \to \infty} f(x) = b$. (*Hint:* Use the arguments used in the Example 2.13.) ◀

2.2 Continuity of a Function

In this section we assume that the domain of a real valued function is an interval I. Recall that every point in an interval I is a limit point of I.

2.2.1 Definition and some basic results

Definition 2.9 Let f be a real valued function defined on an interval I. Then f is is said to be **continuous a point** $x_0 \in I$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad x \in I,\ |x - x_0| < \delta.$$

The function f is said to be **continuous** on I if it is continuous at every $x_0 \in I$. □

Note that if I is an interval and $x_0 \in I$, then x_0 is a limit point of I. Hence, by Theorems 2.3 and 2.4, we can characterize continuity via limits and sequences, as given in the following theorem. Details of its proof is left as an exercise.

Theorem 2.12 *For a function $f : I \to \mathbb{R}$ and $x_0 \in I$, the following are equivalent.*

(i) *f is continuous at x_0.*

(ii) *$\lim_{x \to x_0} f(x)$ exists and it is equal to $f(x_0)$.*

(iii) *For every sequence (x_n) in I with $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.*

CONVENTION: Suppose the domain of a function f is not specified explicitly. Even then we may say that f is continuous at a point $x_0 \in \mathbb{R}$ to mean that f is defined on an interval containing x_0 and f is continuous at x_0.

Example 2.14 Continuity of the functions given in the following examples follows by using the characterization (i) or (ii) in The Theorem 2.12. However, we show how we can use the $\varepsilon - \delta$ arguments to obtain the same conclusions. Let I be an interval.

(i) Every constant function defined on I is continuous: For a give $c \in \mathbb{R}$, let $f(x) = c$, $x \in I$. We may also observe that for any $x_0 \in I$, $|f(x) - f(x_0)| = 0$ so that for any $\varepsilon > 0$,

$$x \in I,\ |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

for any choice of $\delta > 0$.

(ii) Let $f(x) = x$, $x \in I$. Then, for any $x_0 \in I$ we have $|f(x) - f(x_0)| = |x - x_0|$ so that for any $\varepsilon > 0$,

$$x \in I,\ |x - x_0| < \delta := \varepsilon \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Hence f is continuous on I.

(ii) Let $f(x) = x^2$, $x \in I$.

Then, f is continuous on I: For $x_0 \in I$ and $\varepsilon > 0$ be given. We have

$$\begin{aligned} |f(x) - f(x_0)| &= |(x + x_0)(x - x_0)| \\ &\le (|x| + |x_0)(x - x_0)| \\ &\le (|x - x_0| + 2|x_0|)|x - x_0|. \end{aligned}$$

Hence, $|f(x) - f(x_0)| < \varepsilon$ if $(|x - x_0| + 2|x_0|)|x - x_0| < \varepsilon$. Hence, we may choosing $\delta > 0$ such that $(\delta + 2|x_0|)\delta < \varepsilon$, we obtain

$$x \in I,\ |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < (\delta + 2|x_0|)\delta < \varepsilon.$$

For example, we may take $0 < \delta < \min\{1, \varepsilon/(1 + 2|x_0|)\}$.

so that for any $\varepsilon > 0$,

$$x \in I,\ |x - x_0| < \delta := \varepsilon \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Hence f is continuous on I.

□

The following theorem is a consequence of Theorem 2.5 and Theorem 2.12.

Theorem 2.13 *Suppose f and g are defined on an interval I and both f and g are continuous at $x_0 \in I$. Then $f + g$ and fg are continuous at x_0.*

The following Theorem is analogous to Theorem 2.9.

Theorem 2.14 *Suppose $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in I$ and $g : J \to \mathbb{R}$ is continuous at the point $y_0 := f(x_0)$, where J is an interval such that $f(I) \subseteq J$. Then $g \circ f : I \to \mathbb{R}$ is continuous at x_0.*

Proof. Let (x_n) be any sequence in I such that $x_n \to x_0$. Since f is continuous at x_0, we have $f(x_n) \to f(x_0)$. Let $y_n = f(x_n)$, $n \in \mathbb{N}$. Since f is continuous at $y_0 := f(x_0)$, $g(y_n) \to g(y_0)$. Thus, we have proved that for every sequence (x_n) in I with $x_n \to x_0$, $(g \circ f)(x_n) \to (g \circ f)(x_0)$. Hence, $g \circ f$ is continuous at x_0. ∎

The following characterization of continuity at a point is worth noticing.

Theorem 2.15 *A function $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in I$ if and only if for every open interval J containing $f(x_0)$, there exists an open interval I_0 containing x_0 such that*

$$x \in I_0 \cap I \quad \Longrightarrow f(x) \in J.$$

Proof. Suppose f is continuous at x_0 and $J := (\alpha, \beta)$ such that $f(x_0) \in J$. For $\varepsilon > 0$, let $\delta > 0$ be such that

$$x \in I, \quad |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon,$$

i.e., taking $I_0 = (x_0 - \delta, x_0 + \delta)$,

$$x \in I_0 \cap I \quad \Longrightarrow \quad f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon).$$

Choosing $\varepsilon > 0$ such that $\alpha < f(x_0) - \varepsilon$ and $f(x_0) + \varepsilon) < \beta$, i.e.,

$$0 < \varepsilon < \min\{\beta - f(x_0),\, f(x_0) - \alpha\},$$

we obtain

$$x \in I_0 \cap I \quad \Longrightarrow \quad f(x) \in (\alpha, \beta).$$

Conversely, suppose that for every open interval J containing $f(x_0)$, there exists an open interval I_0 containing x_0 such that $x \in I_0 \cap I$ implies $f(x) \in J$. So, given $\varepsilon > 0$, we may take $J = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Let the corresponding I_0 be (a, b). Then with $0 < \delta < \min\{x_0 - a,\, b - x_0\}$, we obtain

$$x \in I, \quad |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Thus, f is continuous at x_0. ∎

Theorem 2.16 *Suppose f is a continuous function defined on an interval I and $x_0 \in I$. Suppose $f(x_0) \neq 0$. Then there exists an open interval I_0 containing x_0 such that $f(x) \neq 0$ for every $x \in I_0 \cap I$. Further, the function $g : I_0 \cap I \to \mathbb{R}$ defined by $g(x) = 1/f(x)$ is continuous at x_0.*

Proof. Let $J = (\alpha, \beta)$ be an open interval containing $f(x_0)$ such that $0 \notin J$. Then by Theorem 2.15, there exists an open interval I_0 containing x_0 such that $f(x) \in J$ whenever $x \in I_0 \cap I$. In particular, $f(x) \neq 0$ for all $x \in I_0 \cap I$ and $g(x) = 1/f(x)$ is defined on $I_0 \cap I$.

Next, we observe that for every $x \in I_0 \cap I$,

$$\frac{1}{f(x)} - \frac{1}{f(x_0)} = \frac{f(x_0) - f(x)}{f(x) f(x_0)}.$$

Since $f(x) \neq 0$ for all $x \in I_0 \cap I$ we have $|f(x)| > c := \min\{|\alpha|, |\beta|\}$ for all $x \in I_0 \cap I$. Therefore,

$$\left|\frac{1}{f(x)} - \frac{1}{f(x_0)}\right| = \frac{|f(x_0) - f(x)|}{|f(x) f(x_0)|} \leq \frac{|f(x_0) - f(x)|}{c^2}$$

for all $x \in I_0 \cap I$. Now, by continuity of f at x_0, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x_0) - f(x)| < c^2\varepsilon \quad \text{whenever} \quad x \in I_0 \cap I, \quad |x - x_0| < \delta.$$

Hence,

$$\left|\frac{1}{f(x)} - \frac{1}{f(x_0)}\right| < \varepsilon \quad \text{whenever} \quad x \in I_0 \cap I, \quad |x - x_0| < \delta.$$

Thus, $1/f$ is continuous at x_0. ∎

Theorems 2.13 and 2.15 imply the following theorem.

Theorem 2.17 *Suppose $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous at a point $x_0 \in I$ and $g(x_0) \neq 0$. Then there exists an open interval I_0 containing x_0 such that f/g is well defined on $I_0 \cap I$ and f/g is continuous at x_0.*

***Exercise* 2.9** Suppose f is a continuous function defined on an interval I and $x_0 \in I$. Prove the following.

1. If $\alpha \geq 0$ is such that $|f(x_0)| > \alpha$, then there exists a subinterval I_0 of I containing x_0 such that $|f(x)| > \alpha$ for all $x \in I_0$.
2. If $f(x_0) > 0$, then then there exists a subinterval I_0 of I containing x_0 such that $|f(x)| \geq f(x_0)/2$ for all $x \in I_0$.
3. If $f(x_0) < 0$, then then there exists a subinterval I_0 of I containing x_0 such that $|f(x)| \leq f(x_0)/2$ for all $x \in I_0$.

◀

2.2.2 Some more examples

In the following examples a particular procedure is adopted to show continuity or discontinuity of a function. The reader may adopt any other alternate procedure, for instance, any one of the characterizations in Theorem 2.12.

Example 2.15 For real numbers $a_0, a_1, \ldots, a_k$, let $f(x) = a_0 + a_1 x + \ldots + a_k x^k$ for $x \in \mathbb{R}$. Since containt functions and the function $f_0(x) = x, x \in \mathbb{R}$ are continuous, by Theorem 2.13, f is continuous on any interval I. □

Example 2.16 For given $x_0 \in \mathbb{R}$, let $f(x) = |x - x_0|$, $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$. To see this, note that, for $a \in \mathbb{R}$,

$$|f(x) - f(a)| = ||x - x_0| - |a - x_0|| \leq |(x - x_0) - (a - x_0)| = |x - a|.$$

Hence, for every $\varepsilon > 0$, we have

$$|x - a| < \varepsilon \Longrightarrow |f(x) - f(a)| < \varepsilon.$$

□

Example 2.17 Let $f(x) = \frac{x^2-4}{x-2}$ for $x \in \mathbb{R}\setminus\{2\}$ and $f(2) = 4$. Then f is continuous on $\mathbb{R}$ (*Verify*). □

Example 2.18 The functions f, g, h defined by

$$f(x) = \sin x, \quad g(x) = \cos x, \quad h(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$$

are continuous on $\mathbb{R}$:

Note that for $x, y \in \mathbb{R}$,

$$\sin x - \sin y = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right)$$

so that

$$|\sin x - \sin y| \leq |x - y| \qquad \forall x, y \in \mathbb{R}.$$

Hence, for every $\varepsilon > 0$ and for every $x_0 \in \mathbb{R}$,

$$x \in \mathbb{R}, \quad |x - x_0| < \varepsilon \quad \Longrightarrow \quad |\sin x - \sin x_0| < \varepsilon.$$

Thus, f is continuous at every point in $\mathbb{R}$. Since $\cos x = 1 - 2\sin^2(x/2)$, $x \in \mathbb{R}$, it also follows that g is also at every point in $\mathbb{R}$. To see the continuity of h on $\mathbb{R}$, first we recall that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Hence, h is continuous at 0. Now, let $x_0 \neq 0$. Then the continuity of h at x_0 follows from Theorem 2.17, since $h = f/f_0$ where $f_0(x) = x$, $x \in \mathbb{R}$.

Continuity of h at a non-zero x_0 is seen directly as follows: Note that, for $x \neq 0$, $x_0 \neq 0$,

$$\begin{aligned} \frac{\sin x}{x} - \frac{\sin x_0}{x_0} &= \frac{x_0 \sin x - x \sin x_0}{x x_0} \\ &= \frac{(x_0 - x)\sin x + x(\sin x - \sin x_0)}{x x_0}. \end{aligned}$$

Hence, using the fact that $|\sin x| \leq |x|$ and $|\sin x - \sin x_0| \leq |x - x_0|$, we have

$$\begin{aligned} \left|\frac{\sin x}{x} - \frac{\sin x_0}{x_0}\right| &\leq \frac{|x_0 - x||\sin x| + |x|\,|\sin x - \sin x_0|}{|x x_0|} \\ &\leq \frac{|x_0 - x||x| + |x|\,|x - x_0|}{|x x_0|} \\ &= \frac{2|x_0 - x|}{|x_0|}. \end{aligned}$$

Thus for a given $\varepsilon > 0$,

$$\left|\frac{\sin x}{x} - \frac{\sin x_0}{x_0}\right| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \varepsilon |x_0|/2.$$

□

Example 2.19 By Theorem 2.16, the function f defined by $f(x) = 1/x$, $x \neq 0$ is continuous at every $x_0 \neq 0$. Recall that the above function f does not have a limit at $x_0 = 0$. Hence, the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ c, & x = 0 \end{cases}$$

is not continuous at $x_0 = 0$ for any given $c \in \mathbb{R}$. □

Example 2.20 Let f be defined by $f(x) = 1/x$ on $(0, 1]$. Then there does not exist a continuous function g on $[0, 1]$ such that $g(x) = f(x)$ for all $x \in (0, 1]$:

Suppose g is any function defined on $[0, 1]$ such that $g(x) = f(x)$ for all $x \in (0, 1]$. Then we have $1/n \to 0$ but $g(1/n) = f(1/n) = n \to \infty$. Thus, $g(1/n) \not\to g(0)$. □

***Exercise* 2.10** Show by $\varepsilon - \delta$ arguments that f defined by $f(x) = 1/x$, $x \neq 0$ is continuous at every $x_0 \neq 0$. ◀

Example 2.21 The function f defined by $f(x) = \sqrt{x}$, $x \geq 0$ is continuous at every $x_0 \geq 0$:

Let $\varepsilon > 0$ be given. First consider the point $x_0 = 0$. Then we have

$$|f(x) - f(x_0)| = \sqrt{x} < \varepsilon \quad \text{whenever} \quad |x| < \varepsilon^2.$$

Thus, f is continuous at $x_0 = 0$. Next assume that $x_0 > 0$. Since $|x - x_0| = (\sqrt{x} + \sqrt{x_0})|\sqrt{x} - \sqrt{x_0}|$, we have

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{|x - x_0|}{\sqrt{x_0}}.$$

Thus,

$$|\sqrt{x} - \sqrt{x_0}| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta := \varepsilon\sqrt{x_0}.$$

□

More generally, we have the following example.

Example 2.22 Let $k \in \mathbb{N}$. Then the function f defined by $f(x) = x^{1/k}$, $x \geq 0$ is continuous at every $x_0 \geq 0$:

Let $\varepsilon > 0$ be given. First consider the point $x_0 = 0$. Then we have

$$|f(x) - f(x_0)| = x^{1/k} < \varepsilon \quad \text{whenever} \quad |x| < \varepsilon^k.$$

Thus, f is continuous at $x_0 = 0$. Next assume that $x_0 > 0$. Let $y = x^{1/k}$ and $y_0 = x_0^{1/k}$. Since

$$y^k - y_0^k = (y - y_0)(y^{k-1} + y^{k-2}y_0 + \ldots + yy^{k-2} + y_0^{k-1}),$$

so that

$$x - x_0 = (x^{1/k} - x_0^{1/k})(y^{k-1} + y^{k-2}y_0 + \ldots + yy^{k-2} + y_0^{k-1}).$$

Hence,

$$|x^{1/k} - x_0^{1/k}| = \frac{|x - x_0|}{y^{k-1} + y^{k-2}y_0 + \ldots + yy^{k-2} + y_0^{k-1}} \le \frac{|x - x_0|}{y_0^{k-1}}.$$

Thus,

$$|x^{1/k} - x_0^{1/k}| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta := \varepsilon y_0^{k-1} = \varepsilon x_0^{1-1/k}.$$

Thus, f is continuous at every $x_0 > 0$. □

Example 2.23 For a rational number r, let $f(x) = x^r$ for $x > 0$. Then using Example 2.22 together with Theorem 2.14, we see that f is continuous at every $x_0 > 0$. □

We know that given $r \in \mathbb{R}$, there exists a sequence (r_n) of rational numbers such that $r_n \to r$. For $n \in \mathbb{N}$, let $f_n(x) = x^{r_n}$, $x > 0$. Since each f_n is continuous for $x > 0$, one may enquire whether the function f defined by $f(x) = x^r$ is continuous for $x > 0$.

First of all how do we define the x^r for $x > 0$?

We shall discuss this issue in a latter section, where we shall introduce two important classes of functions, namely, *exponential* and *logarithm functions*. In fact, our discussion will also include, as special cases, the Examples 2.21 - 2.23.

***Exercise* 2.11** Let I be an interval and $f : I \to \mathbb{R}$. Suppose there exists a constant $K > 0$ such that

$$|f(x) - f(y)| \le K|x - y| \quad \forall\, x, y \in I. \qquad (*)$$

Show that f is continuous on I. Find an example of a continuous function which does not satisfy $(*)$ for any $K > 0$. [*Hint:* Consider $f(x) = \frac{1}{x}$ for $x \in (0, 1]$.]

A function f satisfying $(*)$ for some $K > 0$ is called a *Lipschitz continuous function*, and the constant K called the *Lipschitz constant.* ◀

2.2.3 Some properties of continuous functions

Recall that a subset S of $\mathbb{R}$ is said to be *bounded* if there exists $M > 0$ such that $|s| \le M$ for all $s \in S$, and set which is not bounded is called an *unbounded set.*

Recall that if S is a bounded subset of $\mathbb{R}$, then S has infimum and supremum.

***Exercise* 2.12** Let $S \subseteq \mathbb{R}$. Prove the following:

(i) Suppose S is bounded, and say $\alpha := \inf S$ and $\beta := \sup S$. Then there exist sequences (s_n) and (t_n) in S such that $s_n \to \alpha$ and $t_n \to \beta$.

(ii) S is unbounded if and only if there exists a sequence (s_n) in S which is unbounded.

(iii) S is unbounded if and only if there exists a sequence (s_n) in S such that $|s_n| \to \infty$ as $n \to \infty$.

(iv) If (s_n) is a sequence in S which is unbounded, then there exists a subsequence (s_{k_n}) of (s_n) such that $|s_{k_n}| \to \infty$ as $n \to \infty$.

(v) If (s_n) is a sequence in S such that $|s_n| \to \infty$ as $n \to \infty$, and if (s_{k_n}) is subsequence of (s_n), then $|s_{k_n}| \to \infty$ as $n \to \infty$. ◀

Definition 2.10 A real valued function defined on a set $D \subseteq \mathbb{R}$ is said to be a **bounded function** if the set $\{f(x) : x \in D\}$ is bounded. A function is said to be an **unbounded function** if it is not bounded. □

The following can be easily deduced from the definition:

- A function $f : D \to \mathbb{R}$ is bounded if and only if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in D$.
- A function $f : D \to \mathbb{R}$ is unbounded if and only if there exists a sequence $(x_n) \in D$ such that the $|f(x_n)| \to \infty$ as $n \to \infty$.

Theorem 2.18 *Suppose f is a real valued continuous function defined on a closed and bounded interval $[a, b]$. Then f is a bounded function.*

Proof. Assume for the time being that f is not a bounded function. Then, there exists a sequence (x_n) in $[a, b]$ such that $|f(x_n)| \to \infty$ as $n \to \infty$. Since (x_n) is a bounded sequence, by Bolzano-Weierstrass property of $\mathbb{R}$, there exists a subsequence (x_{k_n}) of (x_n) such that $x_{k_n} \to x$ for some $x \in [a, b]$. Therefore, by continuity of f, $f(x_{k_n}) \to f(x)$. In particular, $(f(x_{k_n}))$ is a bounded sequence. This is a contradiction to the fact that $|f(x_n)| \to \infty$ as $n \to \infty$. Thus, we have proved that f cannot be unbounded. ▮

Remark 2.3 The conditions in Theorem 2.18 are only sufficient condtions; they are not necessary conditions. To see this consider the function

$$f(x) = \begin{cases} 1, & 1 < x \leq 1, \\ 2, & 1 < x < \infty. \end{cases}$$

Then f defined on $I = (1, \infty)$ is not continuous and I is neither closed nor bounded, but f is a bounded function.

It is also true that, if we drop any of the conditions in the theorem, then the conclusion need not be true. To see this consider the unbounded fuctions in the following examples:

1. Let $f(x) = \begin{cases} \frac{1}{x}, & x \in (0, 1], \\ 1, & x = 0. \end{cases}$ In this case f is not continuous, though it is defined on a closed and bounded interval $[0, 1]$.

2. Let $f(x) = \dfrac{1}{x}$, $x \in (0, 1]$. In this case f is is continuous, but its domain $(0, 1]$ is not a closed set.
3. Let $f(x) = x$, $x \in [0, \infty)$. In this case f is is continuous, but its domain $[0, \infty)$ is not bounded.

♦

Attaining $\max f$ **and** $\min f$

Suppose f is a continuous real valued function defined on a closed and bounded interval $[a, b]$. Then, by Theorem 2.18, f is a bounded function. Therefore,

$$\inf_{a \leq x \leq b} f(x) := \inf\{f(x) : x \in [a, b]\}$$

and

$$\sup_{a \leq x \leq b} f(x) := \sup\{f(x) : x \in [a, b]\}$$

exist.

Theorem 2.19 *Suppose f is a continuous function defined on a closed and bounded interval $[a, b]$. Then there exists x_0, y_0 in $[a, b]$ such that*

$$f(x_0) = \inf_{a \leq x \leq b} f(x) \quad \text{and} \quad f(y_0) = \sup_{a \leq x \leq b} f(x).$$

Proof. By the definition of the infimum of a set, there exists a sequence (x_n) in $[a, b]$ such that $f(x_n) \to \alpha := \inf\limits_{a \leq x \leq b} f(x)$. Since (x_n) is a bounded sequence, there exist a subsequence (x_{k_n}) such that $x_{k_n} \to x$ for some $x \in [a, b]$. By continuity of f, $f(x_{k_n}) \to f(x)$. But, we already have $f(x_{k_n}) \to \alpha$. Hence, $\alpha = f(x)$ and $\beta = f(y)$.

Similarly, using the definition of supremum, it can be shown that there exists $y_0 \in [a, b]$ such that $f(y_0) = \sup_{a \leq x \leq b} f(x)$. ∎

The proof of the following corollary is a consequence of Theorem 2.19.

Corollary 2.20 *Suppose f is a continuous function defined on a closed and bounded interval I. Then range of f is a bounded set.*

Remark 2.4 By Theorem 2.19, we say that the infimum and supremum of a continuous real valued function f defined on a closed and bounded interval $[a, b]$ are attained at some points in $[a, b]$, and in that case, we write

$$\inf\{f(x) : x \in [a, b]\} = \min_{a \leq x \leq b} f(x), \qquad \sup\{f(x) : x \in [a, b]\} = \max_{a \leq x \leq b} f(x).$$

The conclusion in the above theorem need not hold if the domain of the function is not of the form $[a, b]$ or if f is not continuous. For example, $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$ for $x \in (0, 1]$ is continuous, but does not attain supremum. Same is the case if $g : [0, 1] \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} \frac{1}{x}, & x \in (0, 1], \\ 1, & x = 0. \end{cases}$$

Thus, neither continuity nor the fact that the domain is a closed and bounded interval can be dropped. This does not mean that the conclusion in the theorem does not hold for all such functions! For example $f : [0, 1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x \in [0, 1/2), \\ 1, & x \in [1/2, 1). \end{cases}$$

Then we see that neither f is continuous, nor its domain of the form $[a, b]$. But, f attains both its maximum and minimum. ♦

Intermediate value theorem

Suppose f is a continuous real valued function defined on a closed and bounded interval $[a, b]$, and

$$\alpha := \min_{a \le x \le b} f(x), \qquad \beta := \max_{a \le x \le b} f(x).$$

Clearly,

$$\alpha \le f(x) \le \beta \qquad \forall\, x \in [a, b].$$

Now, the question is whether every value between α and β is attained by the function. The answer is in affirmative. In fact we have the following general theorem, known as *Intermediate value theorem.*

Theorem 2.21 (Intermediate value theorem (IVT)) *Suppose f is a continuous function defined on an interval I. Suppose x_1 and x_2 are in I such that $f(x_1) < f(x_2)$, and c is such that $f(x_1) < c < f(x_2)$. Then there exists x_0 lying between x_1 and x_2 such that $f(x_0) = c$.*

Before giving its proof, let us look at the interpretations of the theorem geometrically and algebraically.

Geometrically:

> Consider a curves C_1 and C_2 with equations
>
> $$y = f(x) \quad \text{and} \quad y = c,$$
>
> respectively, where $a \le x \le b$ and f is a continuous funciton on $[a, b]$. If c lies between the values $f(a)$ and $f(b)$, then the curves C_1 and C_2 intersect.

Algebraically:

If f is a continuous function on $[a, b]$ and c lies between $f(a)$ and $f(b)$, then the equation

$$f(x) = c$$

has atleast one solution in $[a, b]$.

Proof of Theorem 2.21. Without loss of generality assume that $x_1 < x_2$. Let

$$S = \{x \in [x_1, x_2] : f(x) < c\}.$$

Then S is non-empty (since $x_1 \in S$) and bounded above (since $x \leq x_2$ for all $x \in S$). Let

$$\alpha := \sup S.$$

Then there exists a sequence (a_n) in S such that $a_n \to \alpha$. Note that $\alpha \in [x_1, x_2]$. Hence, by continuity of f, $f(a_n) \to f(\alpha)$. Since $f(a_n) < c$ for all $n \in \mathbb{N}$, we have $f(\alpha) \leq c$. Note that $\alpha \neq x_2$, since $f(\alpha) \leq c < f(x_2)$.

Now, let (b_n) be a sequence in (α, x_2) such that $b_n \to \alpha$. Then, again by continuity of f, $f(b_n) \to f(\alpha)$. Since $b_n > \alpha$, $b_n \notin S$ and hence $f(b_n) \geq c$. Therefore, $f(\alpha) \geq c$. Thus, we have prove that there exists $x_0 := \alpha$ such that $f(x_0) \leq c \leq f(x_0)$ so that $f(x_0) = c$. ∎

Remark 2.5 The proof given above for Theorem 2.21 is taken from the book by Ghorpade and Limaye [3]. ♦

The following two corollaries are immediate consequences of the above theorem.

Corollary 2.22 *Let f be a continuous function defined on an interval. Then range of f is an interval.*

Corollary 2.23 *Suppose f is a continuous real valued function defined on an interval I. If $a, b \in I$ are such that $f(a)$ and $f(b)$ have opposite signs, then there exists $x_0 \in I$ such that $f(x_0) = 0$.*

Now, we derive another important property of continuous functions.

Theorem 2.24 *Suppose f is a continuous function defined on a closed and bounded interval I. Then its range is a closed and bounded interval.*

Proof. We know, by Corollaries 2.20 and 2.22, that range of f is a bounded interval, say J. Hence, it is enough to show that J is a closed set, i.e., J contains all its limit points. For this, let y_0 be a limit point of J. Hence, there exists a sequence (y_n) in J which converges to y_0. let $x_n \in I$ be such that $f(x_n) = y_n$, $n \in \mathbb{N}$. Since I is closed and bounded, (x_n) has a subsequence (x_{k_n}) which converges to some point $x_0 \in I$. By continuity of f, $y_{k_n} = f(x_{k_n}) \to f(x_0)$. Thus, we obtain $y_0 = f(x_0) \in J$. This completes the proof. ∎

2.2.4 Continuity of the inverse of a function

Suppose f is defined on a set $D \subseteq \mathbb{R}$. We may recall the following from elementary set theory:

If f is injective, i.e., one-one, then we know that a function g can be defined on the range $E := f(D)$ of f by $g(y) = x$ for $y \in E$, where $x \in D$ is the unique element in x such that $f(x) = y$. The above function g is called the **inverse** of f. Note that the domain of the inverse of f is the range of f.

By Corollary 2.22, we know that range of a continuous function defined on an interval I is also an interval. Suppose f is also injective. The a natural question one would like to ask is whether its inverse is also continuous. First we answer this question affirmatively by assuming that the domain of the function is closed and bounded.

Theorem 2.25 (Inverse Function Theorem) *Let f be a continuous injective function defined on a closed and bounded interval I. Then its inverse from its range is continuous.*

Proof. Suppose $J = f(I)$, the range of f. Let $y_0 \in J$ and (y_n) be a sequence in J which converges to y_0. Let $x_n = f^{-1}(y_n)$, $n \in \mathbb{N}$ and $x_0 = f^{-1}(y_0)$. We have to show that $x_n \to x_0$.

Suppose, on the contrary, $x_n \not\to x_0$. Then there exists $\varepsilon_0 > 0$ and a subsequence (u_n) of (x_n) such that $u_n \notin (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$ for all $n \in \mathbb{N}$. Since I is a bounded interval, (u_n) is a bounded sequence. Hence, (u_n) has a subsequence (v_n) which converges to some $v \in \mathbb{R}$. Since I is a closed interval, $v \in I$. Now, continuity of f implies that $f(v_n) \to f(v)$. But, since $(f(v_n))$ is a subsequence of (y_n), and since $y_n \to y_0$, we have $f(v) = y_0 = f(x_0)$. Now, since f is injective, $v = x_0$. Thus we have proved that $v_n \to x_0$. This is a contradiction to the fact that $v_n \notin (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$ for all $n \in \mathbb{N}$. ∎

Next we shall prove the conclusion in the last theorem by dropping the condition that I is closed and bounded, but assuming an additional condition on f, namely that it is *strictly monotonic.*

So, we have to define what *strict monotonicity* of f is.

Definition 2.11 Let f be defined on an interval I. Then f is said to be

(i) **monotonically increasing** on I if

$$x, y \in I, \quad x < y \Longrightarrow f(x) \leq f(y),$$

(ii) **strictly monotonically increasing** on I if

$$x, y \in I, \quad x < y \Longrightarrow f(x) < f(y),$$

(iii) **monotonically decreasing** on I if

$$x, y \in I, \quad x < y \Longrightarrow f(x) \geq f(y).$$

(iv) **strictly monotonically decreasing** on I if

$$x, y \in I, \quad x < y \Longrightarrow f(x) > f(y).$$

If f is either monotonically increasing (respectively, strictly monotonically increasing) or monotonically decreasing (respectively, strictly monotonically decreasing) on I, then it is called a **monotonic (respectively, strictly monotonic)** function. □

We observe that

- f strictly monotonic on $I \quad \Longrightarrow \quad f$ is injective on I.

The converse of the above statement is true. For example, the function

$$f(x) = \begin{cases} x, & -1 \leq x \leq 0, \\ 1 - x, & 0 < x \leq 1, \end{cases}$$

is injective but not strictly monotonic on $[-1, 1]$.

Sometimes, the terminology increasing , decreasing, strictly increasing, strictly decreasing, are used in place of monotonically increasing, monotonically decreasing, strictly monotonically increasing, and strictly monotonically decreasing, respectively.

Example 2.24 We observe the following.

(i) The function $f(x) = x$ is strictly increasing on $\mathbb{R}$.

(ii) The function $f(x) = -x$ is strictly increasing on $\mathbb{R}$.

(iii) The function $f(x) = x^2$ is strictly increasing for $x \geq 0$ and strictly decreasing for $x \leq 0$.

(iv) The function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$.

(vi) The function $f(x) = \sin x$ is strictly increasing on $[0, \pi/2]$ and strictly decreasing on $[\pi/2, \pi]$.

(vi) The function $f(x) = \cos x$ is strictly decreasing on $[0, \pi]$. □

Theorem 2.26 (Inverse Function Theorem) *Let f be a continuous function defined on an interval I. Suppose f is strictly monotonic on I. Then f is injective and its inverse from its range is continuous.*

Proof. We assume that f is strictly monotonically increasing. The case when strictly monotonically increasing will follows by similar arguments.

Since f is continuous, its range is also an interval, say J. By the assumption, for $x_1, x_2 \in J$, $x_1 < x_2 \implies f(x_1) < f(x_2)$. Hence, f is injective. Let g be its inverse from the range J. Let $y_0 \in J$ and (y_n) in J be such that $y_n \to y_0$. Let $x_n = g(y_n)$, $n \in \mathbb{N}$ and $x_0 = g(y_0)$. We have to show that $x_n \to x_0$. Suppose $x_n \not\to x_0$. Then there exists $\varepsilon > 0$ and a subsequence (x_{k_n}) of (x_n) such that $|x_{k_n} - x_0| \geq \varepsilon$, i.e.,

$$x_{k_n} \notin (x_0 - \varepsilon, x_0 + \varepsilon) \quad \forall\, n \in \mathbb{N}.$$

Note that, at the moment, we cannot write $f(x_{k_n}) \notin (f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ for all $n \in \mathbb{N}$ so as to conclude that $y_{k_n} \not\to 0$ and thus arrive at a contradiction, because we do not know that $x_0 - \varepsilon$ and $x_0 + \varepsilon$ belong to the domain of f. So, we consider the following three mutually exclusive cases:

(i) $x_{k_n} \leq x_0 - \varepsilon \quad \forall\, n \in \mathbb{N}$,

(ii) $x_{k_n} \geq x_0 + \varepsilon \quad \forall\, n \in \mathbb{N}$,

(iii) $\exists\, n, m \in \mathbb{N}$ such that $x_{k_n} \leq x_0 - \varepsilon$ and $x_{k_m} \geq x_0 + \varepsilon$.

Since $x_0 \in I$ and $x_{k_n} \in I$ for all $n \in \mathbb{N}$, in case (i), $[x_0 - \varepsilon, x_0] \subseteq I$, in case (ii), $[x_0, x_0 + \varepsilon] \subseteq I$, and in case (iii), $[x_0 - \varepsilon, x_0 + \varepsilon] \subseteq I$. Thus, by strict monotonicity of f, we have

(a) $x_0 - \varepsilon \in I$ and $y_{k_n} \leq f(x_0 - \varepsilon) < y_0 \quad \forall\, n \in \mathbb{N}$,

(b) $x_0 + \varepsilon \in I$ and $y_0 < f(x_0 + \varepsilon) \leq y_{k_n} \quad \forall\, n \in \mathbb{N}$,

(c) $x_0 - \varepsilon,\ x_0 + \varepsilon \in I$ and $y_{k_n} \notin (f(x_0 - \varepsilon),\ f(x_0 + \varepsilon)) \quad \forall\, n \in \mathbb{N}$

in cases (i), (ii), (iii), respectively. Hence, we can conclude that $y_{k_n} \not\to y_0$, which is a contradiction. Thus, we have proved that g is continuous. ∎

Remark 2.6 (i) In the proof of Theorem 2.26, the continuity of f is used only to assert that its range J is an interval so that its inverse f^{-1} is defined on an interval.

(ii) We know that strict monotonicity of a function implies that it is injective, but injectivity does not implies strict monotonicity. So, one may ask whether strict monotonicity assumption in Theorem 2.26 can be replaced by injectivity. The answer is in affirmative as the following Exercise shows. ♦

***Exercise* 2.13** Let f be an injective function defined on an interval I. Show that if f is continuous, then it is strictly monotonic on I [Hint: Use Intermediate Value Theorem]. ◄

2.2.5 Exponential and logarithm functions

We have already come across expression such as a^b for $a > 0$ and $b \in \mathbb{R}$, though we have not proved its existence. Also we have seen that

(i) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{1/n}$ exists,

(ii) $\sum_{n=0}^{\infty}\frac{1}{n!}$ converges,

and they are same, and denoted the common value by e (after Euler). We have also shown that

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{1/x}.$$

From elementary arithmetic we know that for $m, n \in \mathbb{N}$, $e^{m+n} = e^m e^n$, and e^n is defined by $e^{-n} = \frac{1}{e^n}$. Thus, using the convention $e^0 = 1$, we have

$$e^{m+n} = e^m e^n \quad \forall\, m, n \in \mathbb{Z}.$$

For $n \in \mathbb{N}$, we may define $e^{1/n}$ as the n^{th} root of e. Once this is done we can define e^r for any rational number r. But, proof of the existence of the n^{th} root of a positive number is quite involved. We shall consider an alternate method for proving the same thing, by using the concept of an *exponential function* $\exp(x)$, $x \in \mathbb{R}$. First, we observe that the series

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

converges absolutely for every $x \in \mathbb{R}$. This is easily seen by using the ratio test. This series plays a very significant role in mathematics.

Definition 2.12 For $x \in \mathbb{R}$, the function

$$\exp(x) := \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad x \in \mathbb{R},$$

is called the **exponential function**. $\square$

Clearly,

$$\exp(0) = 1, \quad \exp(1) = e.$$

Our first attempt is to show that

$$\exp(r) = e^r$$

for every rational number. In order to do that we have to derive some of the important properties of the function $\exp(x)$. For that purpose, first we observe the following result on convergence of series.

Theorem 2.27 *Suppose that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent series, and*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n \in \mathbb{N}.$$

Then, the series $\sum_{n=0}^{\infty} c_n$ converges absolutely and

$$\Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big) = \sum_{n=0}^{\infty} c_n.$$

Proof. Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergence, they are convergent. Let their sums be A and B, respectively. Let

$$A_n = \sum_{i=0}^{n} a_i, \quad B_n = \sum_{i=0}^{n} b_i, \quad C_n = \sum_{i=0}^{n} c_i.$$

Then $A_n \to A$, $B_n \to B$ and $A_n B_n \to AB$. We have to prove that $C_n \to AB$.

First let us assume that the terms of the series are with positive terms. Note that, if

$$\alpha_{ij} = a_i b_j, \quad i, j = 0, 1, \ldots, n,$$

then $A_n B_n$ is the sum of all entries of the matrix (α_{ij}) and C_n is the sum of the entries of the left upper triangular part of the matrix (α_{ij}), i.e.,

$$A_n B_n = \sum_{i=0}^{n}\sum_{j=0}^{n} \alpha_{ij}, \quad C_n = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \alpha_{ij}.$$

Hence, it follows that

$$C_n \le A_n B_n \le C_{2n} \tag{1}$$

for all $n \in \mathbb{N}$. Since $(A_n B_n)$ converges to AB and (C_n) is an increasing sequence of nonnegative terms, the relation (1) implies that (C_n) is bounded, and hence it converges. Let $C_n \to C$. Again, (1) together with sandwich theorem implies that $C_n \to AB$. This proves the case when the series are with nonnegative terms.

Next let us consider the general case. By what we have proved in last paragraph, we have

$$\Big(\sum_{i=0}^{\infty} |a_i|\Big)\Big(\sum_{i=0}^{\infty} |b_i|\Big) = \sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k} |a_i|\,|b_{k-i}|\Big).$$

Let

$$\hat{A}_n = \sum_{i=0}^{n} |a_i|, \quad \hat{B}_n = \sum_{i=0}^{n} |b_i|, \quad D_n = \sum_{k=0}^{n}\Big(\sum_{i=0}^{k} |a_i|\,|b_{k-i}|\Big).$$

As in last paragraph, we obtain

$$D_n \le \hat{A}_n \hat{B}_n \le D_{2n}$$

so that

$$|A_nB_n - C_n| \le \hat{A}_n\hat{B}_n - D_n \le D_{2n} - D_n. \qquad (2)$$

Since (D_n) converges, we obtain $D_{2n} - D_n \to 0$, and since $A_nB_n \to AB$, we have the convergence $C_n \to AB$. ▮

Definition 2.13 The series $\sum_{n=0}^{\infty} c_n$ with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ is called the **Cauchy product** of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. □

Now, we observe some properties of $\exp(\cdot)$.

Theorem 2.28 *Let* $\exp(\cdot)$ *be the function as in Definition 2.12. Then the following results hold.*

(i) $\exp(x + y) = \exp(x)\exp(y) \quad \forall\, x, y \in \mathbb{R}$

(ii) $\exp(x) \neq 0 \quad \forall\, x \in \mathbb{R}$.

(iii) $\exp(-x) = \dfrac{1}{\exp(x)} \quad \forall\, x \in \mathbb{R}$.

(iv) $\exp(x) > 0 \quad \forall\, x \in \mathbb{R}$.

(v) $\exp(kx) = [\exp(x)]^k \quad \forall\, x \in \mathbb{R},\ k \in \mathbb{Z}$. *In particular,*

  (a) $\exp(k) = e^k, \quad \forall\, k \in \mathbb{Z}$,

  (b) $[\exp(1/k)]^k = e \quad \forall\, k \in \mathbb{Z}$.

  (c) $\exp(m/n) = [\exp(1/n)]^m \quad \forall\, m, n \in \mathbb{Z}$ *with* $n \neq 0$.

(vi) $\exp(x) > 1 \iff x > 0$ *and* $\exp(x) = 1 \iff x = 0$.

(vii) $x > y \iff \exp(x) > \exp(y)$.

(viii) $\exp(x) \to \infty$ *as* $x \to \infty$.

(ix) $\exp(x) \to 0$ *as* $x \to -\infty$.

Proof. Note that, for $x, y \in \mathbb{R}$,

$$\frac{(x+y)^n}{n!} = \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k} = \sum_{k=0}^{n} \frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!}.$$

Hence, by Theorem 2.27 by taking $a_n = x^n/n!$ and $b_n = y^n/n!$, we have

$$\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \Big(\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big)\Big(\sum_{n=0}^{\infty} \frac{y^n}{n!}\Big).$$

This proves (i). The results in (ii) and (iii) follow from (i), and the result in (iv) follows from (iii), since $\exp(x) > 0$ for $x \geq 0$, and (v) follows from (i).

To see (vi), observe that $x > 0$ implies $\exp(x) > 1$. Next, suppose $x \leq 0$. If $x = 0$, then $\exp(x) = \exp(0) = 1$. If $x < 1$, then taking $y = -x$, we have $y > 1$, and hence from the first part, $\exp(y) > 1$, i.e., $1/\exp(x) = \exp(-x) > 1$ so that $\exp(x) < 1$. Hence, $\exp(x) > 1 \iff x > 0$. From the above arguments, we also obtain $\exp(x) = 1 \iff x = 0$.

The result in (vii) follows from the facts that

$$x > y \iff x - y > 0 \iff \exp(x - y) > 1$$

and the relation $\exp(x-y) = \exp(x)/\exp(y)$, which is a consequence of (i) and (iii).

The result in (viii) follows from the relation

$$\exp(x) = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} \geq 1 + x \quad \forall\, x > 0,$$

and (ix) is a consequence of (iii) and (viii). ∎

In view of (v)(b) above, we may define

$$e^{1/k} := \exp(1/k) \quad \forall\, k \in \mathbb{N},$$

and hence by (v)(c),

$$e^{m/n} := [e^{1/n}]^m \quad \forall\, m, n \in \mathbb{N}.$$

Thus, for every rational number r, we can define

$$e^r := \exp(r)$$

which satisfies the usual index laws.

We know that every real number is a limit of a sequence of rational numbers. Thus, if $x \in \mathbb{R}$, there exists a sequence (x_n) of rational numbers that $x_n \to x$. So, we may define

$$e^x = \lim_{n\to\infty} e^{x_n}$$

provided the above limit exists. Thus, our next attempt is to show that the function $\exp(x)$, $x \in \mathbb{R}$, is continuous.

Theorem 2.29 *The function* $\exp(\cdot)$ *is continuous on* $\mathbb{R}$

Proof. For brevity of expression, let us use the notation e^x for $\exp(x)$. Let $x, x_0 \in R$. Then we have

$$e^x - e^{x_0} = e^{x_0}(e^{x-x_0} - 1) = e^{x_0} \sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n!} = e^{x_0}(x - x_0) \sum_{n=1}^{\infty} \frac{(x-x_0)^{n-1}}{n!}.$$

Thus, if $|x - x_0| \le 1$, then

$$|e^x - e^{x_0}| \le e^{x_0}|x - x_0| \sum_{n=1}^{\infty} \frac{1}{n!} = e^{x_0}(e-1)|x - x_0|.$$

Hence, for every $\varepsilon > 0$,

$$|e^x - e^{x_0}| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \min\{1, \varepsilon/[e^{x_0}(e-1)]\}$$

so that e^x is a continuous function for $x \in \mathbb{R}$. ▮

NOTATION: We know that for every $x \in \mathbb{R}$, there exists a sequence (x_n) of rational numbers such that $x_n \to x$. In view of Theorem 2.29,

$$e^{x_n} = \exp(x_n) \to \exp(x).$$

Hence, we shall use the notation e^x for $\exp(x)$ for every $x \in \mathbb{R}$. With this notation we have the following identity:

$$e^{x+y} = e^x e^y \quad \forall\, x, y \in \mathbb{R}.$$

Theorem 2.30 *The function e^x is bijective from $\mathbb{R}$ to $(0, \infty)$.*

Proof. First we observe that, for x_1, x_2 in $\mathbb{R}$

$$e^{x_2} - e^{x_1} = e^{x_1}[e^{x_2 - x_1} - 1].$$

Thus,

$$e^{x_2} = e^{x_1} \iff e^{x_2 - x_1} = 1 \iff x_1 = x_2,$$

showing that the function $x \mapsto e^x$ is one-one.

Next, we show that the function is onto, let $y \in (0, \infty)$. Recall that

$$e^x \to 0 \quad \text{as} \quad x \to -\infty, \qquad e^x \to \infty \quad \text{as} \quad x \to \infty.$$

Hence, there exists $M_1 > 0$ such that $e^x > y$ for all $x > M_1$, and there exists $M_2 > 0$ such that $e^x < y$ for all $x < -M_2$. Now, taking $x_1 > M_1$ and $x_2 < -M_2$, we obtain

$$e^{x_1} < y < e^{x_2}.$$

Hence, by the intermediate value property, there exists $x \in \mathbb{R}$ such that $e^x = y$. ▮

Definition 2.14 For $b > 0$, the unique $a \in \mathbb{R}$ such that $e^a = b$ is called the **natural logarithm** of b, and it is denoted by $\ln b$. The function

$$\ln x, \qquad x > 0,$$

is called the **natural logarithm function**. □

Definition 2.15 For $a > 0$ and $b \in \mathbb{R}$, we define

$$a^b := e^{b \ln a}.$$

□

Remark 2.7 We note that $\ln e = 1$ so that if $a = e$, then the Definition 2.15 matches with Definition 2.12. ♦

Theorem 2.31 *Let $a > 0$. Then the function a^x is continuous and bijective from $\mathbb{R}$ to $(0, \infty)$.*

Proof. Note that for $x \in \mathbb{R}$, $a^x := e^{x \ln a}$. Hence, the result is a consequence of Theorems 2.29 and 2.30, and the Definition 2.15, and using the fact that composition of two continuous functions is continuous. ∎

Definition 2.16 Let $a > 0$. For $c > 0$, the unique $b \in \mathbb{R}$ such that $a^b = c$ is called the **logarithm** of c to the base a, and it is denoted by $\log_a c$. The function

$$\log_a x, \qquad x > 0,$$

is called the **logarithm function**. □

We observe that following.

- For $y \in \mathbb{R}$, $\quad y = \ln x \iff e^y = x$.
- For $a > 0$ and $y \in \mathbb{R}$, $\quad y = \log_a x \iff a^y = x$.
- For $a > 0$ and $x > 0$, $\quad \log_a x = \dfrac{\ln x}{\ln a}$.

***Exercise* 2.14** For $a > 0, b > 0$, show that $(\log_b a)(\log_a b) = 1$. ◀

Theorem 2.32 *The functions $\ln x$ and $\log_a x$ for $a > 0$ are continuous on $(0, \infty)$.*

Proof. Let x, x_0 belong to the interval $(0, \infty)$, and let $y = \ln x$ and $y_0 = \ln x_0$. Then we have $e^y = x$ and $e^{y_0} = x_0$. Assume, without loss of generality that $x > x_0$. Since $e^a > 1$ if and only if $a > 0$, we have $y > y_0$, and hence

$$x - x_0 = e^y - e^{y_0} = e^{y_0}(e^{y - y_0} - 1) = e^{y_0} \sum_{n=1}^{\infty} \frac{(y - y_0)^n}{n!} \geq e^{y_0}(y - y_0).$$

Hence,

$$|y - y_0| \leq e^{-y_0}|x - x_0|.$$

Thus, for $\varepsilon > 0$, we have $|y - y_0| < \varepsilon$ whenever $|x - x_0| < e^{y_0}\varepsilon$, $\ln x$ is continuous on $(0, \infty)$. Since $\log_a x = \ln x / \ln a$, the function $\log_a x$ is also continuous on $(0, \infty)$. ∎

Theorem 2.33 *For $r \in \mathbb{R}$, the function $f : (0,\infty) \to \mathbb{R}$ be defined by*

$$f(x) = x^r, \qquad x \in (0,\infty)$$

is continuous.

Proof. For $r \in \mathbb{R}$ and $x > 0$, we have $x^r = e^{r \ln x}$. Hence, the result follows from Theorem 2.32 and Theorem 2.14. ∎

NOTATION: Often, the notation $\log x$ is used for the natural logarithm function in place $\ln x$.

2.3 Differentiability of functions

2.3.1 Definition and examples

Definition 2.17 Suppose f is a (real valued) function defined on an open interval I and $x_0 \in I$. Then f is said to be **differentiable at** x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and in that case the value of the limit is called the **derivative** of f at x_0.

The derivative of f at x_0, if exists, is denoted by

$$f'(x_0) \quad \text{or} \quad \frac{df}{dx}(x_0)$$

or sometimes

$$\frac{d}{dx} f(x)|_{x=x_0}.$$

□

Remark 2.8 The notation $\dfrac{df}{dx}(x)$, introduced by Leibniz[1], is useful in realizing that the expression $\dfrac{d}{dx}$ is an *operator* which associates each function f differentiable in an open interval I to the function $f'(x)$. ♦

Let f be a real valued function defined on an open interval I containing x_0. We observe the following.

[1]Gottfried Wilhelm von Leibniz (July 1, 1646 November 14, 1716) was a German Mathematician and Philosopher, who was the one of the two founders of Calculus, the other was Isaac Newton (25 December 1642 20 March 1727), the English Physicist and Mathematician.

1. $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ if and only if $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists, and in that case

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

2. $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ if and only if for every sequence (x_n) in $I \setminus \{x_0\}$, $x_n \to x_0$ implies $\lim_{h\to 0} \frac{f(x_n)-f(x_0)}{x_n - x_0}$ exists, and in that case

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_n)-f(x_0)}{x_n - x_0}.$$

CONVENTION: Whenever we say that "a function f is differentiable at a point x_0", we mean that f is a real valued function defined on an open interval I containing x_0 and $f : I \to \mathbb{R}$ is differentiable at x_0.

Example 2.25 Let us look at the following simple examples.

(i) For $c \in \mathbb{R}$, let $f(x) = c, x \in \mathbb{R}$. Then it is clear that for any $x_0 \in \mathbb{R}$,

$$\frac{f(x)-f(x_0)}{x-x_0} = 0 \quad \forall\, x \neq x_0.$$

Hence $f'(x_0) = 0$.

(ii) Let $f(x) = x,\ x \in \mathbb{R}$. Then for any $x_0 \in \mathbb{R}$,

$$\frac{f(x)-f(x_0)}{x-x_0} = 1 \quad \forall\, x \neq x_0.$$

Hence $f'(x_0) = 1$.

(iii) Let $f(x) = \sin x,\ x \in \mathbb{R}$. Then for any $x, x_0 \in \mathbb{R}$ with $x \neq x_0$,

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{2\cos\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x-x_0} = \cos\left(\frac{x+x_0}{2}\right)\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}.$$

Thus we see that $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \cos x_0$ so that $f'(x_0) = \cos x_0$.

(iv) The function e^x is differentiable at ever $x \in \mathbb{R}$ and

$$(e^x)' = e^x \qquad \forall\, x \in \mathbb{R}.$$

To see this, first we note that for $h \neq 0$,

$$\frac{e^{x+h}-e^x}{h} - e^x = \frac{e^x}{h}(e^h - 1 - h) = \frac{e^x}{h}\sum_{n=2}^{\infty}\frac{h^n}{n!} = e^x h \sum_{n=2}^{\infty}\frac{h^{n-2}}{n!}.$$

Hence,

$$|h| \le 1 \Longrightarrow \left| \frac{e^{x+h} - e^x}{h} - e^x \right| \le e^x |h| \sum_{n=2}^{\infty} \frac{1}{n!} = e^x |h|(e-2).$$

From this we obtain that e^x is differentiable at x and its derivative is e^x. □

Remark 2.9 In deriving the result Example 2.25 (iv), we used the following fact: If (a_n) is a sequence such that $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges and

$$\Big| \sum_{n=1}^{\infty} a_n \Big| \le \sum_{n=1}^{\infty} |a_n|.$$

♦

Many of the functions that occur in mathematics can be constructed with the help of the functions considered in the Example 2.25 using some properties of differentiation considered in the next section.

***Exercise* 2.15** Suppose f is defined on an open interval I and $x_0 \in I$. Show that f is differentiable at $x_0 \in I$ if and only if there exists a continuous function $\Phi(x)$ such that

$$f(x) = f(x_0) + \Phi(x)(x - x_0),$$

and in that case $\Phi(x_0) = f'(x_0)$. ◀

***Exercise* 2.16** Let Φ be as in Exercise 2.15. Then f is differentiable at x_0, if and only if for every sequence (x_n) in $I \setminus \{x_0\}$ which converges to x_0, the sequence $\Phi(x_n)$ converges, and in that case $f'(x_0) = \lim_{n\to\infty} \Phi(x_n)$. ◀

2.3.2 Left and right derivatives

Recall that in the definition of continuity of a function we considered the domain of the function to be an interval, not necessarily an open interval, whereas in the definition of differentiability we took the interval to be an open interval. Even in the definition of differentiability we could have taken an arbitrary interval and x_0 can be an end point of I if belongs to that interval. In such case, we have the the so called *right differentiability* or *left differentiability* at x_0 depending on whether x_0 is a right end point or left end point of I.

In fact right differentiability and left differentiability can be defined at an interior point as well. By **interior point of an interval** I we mean those points in I which are not the endpoints. More generally, a point $a \in \mathbb{R}$ is said to be an **interior point** of a set $D \subseteq \mathbb{R}$ if D contains a δ-neighbourhood of a.

Definition 2.18 Let f be a real valued function defined on an interval I and $x_0 \in I$.

1. Let x_0 be a right endpoint or an interior point of I. Then f is said to be **left differentiable** at x_0 if

$$\lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists,}$$

and in that case the above limit is called the **left derivative** of f at x_0, and it is denoted $f'_-(x_0)$.

2. Let x_0 be a left endpoint or an interior point of I. Then f is said to be **right differentiable** at x_0 if

$$\lim_{x\to x_0+} \frac{f(x)-f(x_0)}{x-x_0} \text{ exists,}$$

and in that case the above limit is called the **right derivative** of f at x_0, and it is denoted $f'_+(x_0)$.

□

Remark 2.10 In some of the books in calculus, one may find the notations $f'(x_0-)$ and $f'(x_0+)$ for left derivative and right derivative, respectively, at x_0. We preferred to use the notations $f'_-(x_0)$ and $f'_+(x_0)$ as the notations $f'(x_0-)$ and $f'(x_0+)$ can be confused with the left and right limits of the function f' at the point x_0. Thus, in our notation,

$$f'_-(x_0) := \lim_{x\to x_0-} \frac{f(x)-f(x_0)}{x-x_0}, \qquad f'_+(x_0) := \lim_{x\to x_0+} \frac{f(x)-f(x_0)}{x-x_0}$$

whenever the above limits exists.

♦

The following characterization will help us in checking the existence of left and right derivatives.

(i) Let x_0 be the right endpoint or an interior point of I and $\delta_0 > 0$ be such that $(x_0-\delta_0, x_0] \subseteq I$. Let

$$\Phi_-(x) = \frac{f(x)-f(x_0)}{x-x_0}, \quad x_0-\delta_0 < x < x_0.$$

Then $f'_-(x_0)$ exists if and only if $\lim_{x\to x_0} \Phi_-(x)$ exists, and $f'_-(x_0) = \lim_{x\to x_0} \Phi_-(x)$.

(ii) Let x_0 be the left endpoint or an interior point of I and $\delta_0 > 0$ be such that $[x_0, x_0+\delta_0) \subseteq I$. Let

$$\Phi_+(x) = \frac{f(x)-f(x_0)}{x-x_0}, \quad x_0 < x < x_0+\delta_0.$$

Then $f'_+(x_0)$ exists if and only if $\lim_{x\to x_0} \Phi_+(x)$ exists, and $f'_+(x_0) = \lim_{x\to x_0} \Phi_+(x)$.

The following characterizations are in terms of sequences (*Verify*):

(i) Let x_0 be a right endpoint or an interior point of I. Then $f_-(x_0)$ exists if and only if for every sequence (x_n) in I with $x_n < x_0$ for all $n \in \mathbb{N}$, $x_n \to x_0$ implies $\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}$ exists,, and in that case

$$f'_-(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}.$$

(ii) Let x_0 be a left endpoint or an interior point of I. Then $f_+(x_0)$ exists if and only if for every sequence (x_n) in I with $x_n > x_0$ for all $n \in \mathbb{N}$, $x_n \to x_0$ implies $\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}$ exists, and in that case

$$f'_+(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}.$$

In view of the above discussion, we have the following:

- If x_0 is an interior point of I, then $f'(x_0)$ exists if and only if $f'_+(x_0)$ and $f'_-(x_0)$ exists and $f'(x_0) = f'_+(x_0) = f'_-(x_0)$.

***Exercise* 2.17** Prove the above statement. ◀

Example 2.26 Let

$$f(x) = \begin{cases} 0, & x \in [-1, 0), \\ 1, & x \in [0, 1]. \end{cases}$$

Then f is

1. differentiable at every point in $x_0 \in (-1, 0) \cup (0, 1)$, and $f'(x_0) = 0$,
2. right differentiable at -1 and 0, and $f'_+(-1) = 0$, $f'_+(0) = 0$,
3. left differentiable at 1, and $f'_-(0) = 0$.
4. not left differentiable at 0.

In fact, it can be easily seen that

(i) $x_0 \in (-1, 0) \cup (0, 1) \implies \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = 0$,

(ii) $x_0 = -1 \implies \lim_{x\to x_0+} \frac{f(x) - f(x_0)}{x - x_0} = 0$,

(iii) $x_0 = 0 \implies \lim_{x\to x_0+} \frac{f(x) - f(x_0)}{x - x_0} = 0$ and $\lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0}$ does not exist,

(iv) $x_0 = 1 \implies \lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0} = 0$. □

Example 2.27 Consider the **signum function**, $f(x) = \mathrm{sgn}(x)$, $x \in \mathbb{R}$, that is, $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x/|x| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that $f(x) = 1$ for $x > 0$, $f(x) = -1$ for $x < 0$. Hence, we obtain $f'(x) = 0$ for every $x \neq 0$. Note that

$$\frac{f(x) - f(0)}{x} = \begin{cases} 1/x, & x > 0 \\ -1/x, & x < 0. \end{cases}$$

Hence, neither $f'_+(0)$ nor $f'_-(0)$ exists. □

Example 2.28 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 - |x| & \text{if } x \in [-1, 1], \\ 0 & \text{if } x \notin [-1, 1]. \end{cases}$$

Then we have $f(x) = \begin{cases} 1 - x & \text{if } x \in [0, 1], \\ 1 + x & \text{if } x \in [-1, 0), \\ 0 & \text{if } x \notin [-1, 1]. \end{cases}$ Clearly, f is differentiable at every $x_0 \notin \{-1, 0, 1\}$. Let us consider the situations at the points -1, 0, 1.

(i) $x < -1 \implies \dfrac{f(x) - f(-1)}{x - (-1)} = \dfrac{0 - 0}{x + 1} = 0$. Hence, $f'_-(-1) = 0$.

(ii) $-1 < x < 0 \implies \dfrac{f(x) - f(-1)}{x - (-1)} = \dfrac{(1 + x) - 0}{x + 1} = 1$. Hence, $f'_+(-1) = 1$.

(iii) $-1 < x < 0 \implies \dfrac{f(x) - f(0)}{x} = \dfrac{(1 + x) - 1}{x} = 1$. Hence, $f_-(0) = 1$.

(iv) $0 < x < 1 \implies \dfrac{f(x) - f(0)}{x - (-1)} = \dfrac{(1 - x) - 1}{x} = -1$. Hence, $f'_+(0) = -1$.

(v) $0 < x < 1 \implies \dfrac{f(x) - f(1)}{x - 1} = \dfrac{(1 - x) - 0}{x - 1} = -1$. Hence, $f'_-(1) = -1$.

(vi) $x > 1 \implies \dfrac{f(x) - f(1)}{x - 1} = \dfrac{0 - 0}{x - 1} = 0$. Hence, $f'_+(1) = 0$.

Thus left and right derivatives of f at the points -1, 0, 1 exist, but f is not differentiable at any of these points. □

2.3.3 Some properties of differentiable functions

The proof of the following theorem is easy and hence it is left as an exercise.

Theorem 2.34 *Suppose f and g defined on I are differentiable at a point x_0 and $\alpha \in \mathbb{R}$. Then the functions $f + g$ and αf, defined by*

$$(f + g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x), \quad x \in I,$$

are differentiable at x_0*, and*

$$(f+g)'(x_0) = f'(x_0) + g'(x_0), \qquad (\alpha f)'(x_0) = \alpha\varphi'(x_0).$$

Here is a necessary condition for differentiability.

Theorem 2.35 (Differentiability implies continuity) *Suppose* f *defined at point* x_0*. Then* f *is continuous at* x_0*.*

Proof. Note that

$$f(x) - f(x_0) = \left[\frac{f(x) - f(x_0)}{x - x_0}\right](x - x_0) \to f'(x_0).0 = 0 \quad \text{as} \quad x \to x_0.$$

Thus, f is continuous at x_0. ∎

For the following theorem, we may recall that if a function g is continuous at a point x_0 and $g(x_0) \neq 0$, then there exists an open interval I_0 containing x_0 such that $g(x) \neq 0$ for all $x \in I_0$.

Theorem 2.36 (Products and quotient rules) *Suppose* f *and* g *are differentiable at* x_0*. Then the function* $\varphi(x) := f(x)g(x)$ *is differentiable at* x_0*, and*

$$\varphi'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0). \qquad (*)$$

If $g(x)$ *is nonzero in a neighbourhood of* x_0*, then the function* $\psi(x) := f(x)/g(x)$ *defined in that neighbourhood is differentiable at* x_0*, and*

$$\psi'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}. \qquad (**)$$

Proof. Note that

$$\begin{aligned} \varphi(x) - \varphi(x_0) &= f(x)g(x) - f(x_0)g(x_0) \\ &= [f(x) - f(x_0)]g(x) + f(x_0)[g(x) - g(x)] \end{aligned}$$

so that, using the facts that $f'(x_0)$ and $g'(x_0)$ exist and g is continuous at x_0, obtain

$$\begin{aligned} \frac{\varphi(x) - \varphi(x_0)}{x - x_0} &= \frac{f(x) - f(x_0)}{x - x_0}g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0} \\ &\to f'(x_0)g(x_0) + f(x_0)g'(x_0) \quad \text{as} \quad h \to 0. \end{aligned}$$

Hence, φ is differentiable at x_0, and

$$\varphi'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Also, since

$$\begin{aligned}\psi(x)-\psi(x_0) &= \frac{f(x)g(x_0)-f(x_0)g(x)}{g(x)g(x_0)}\\ &= \frac{[f(x)-f(x_0)]g(x_0)-f(x_0)[g(x)-g(x_0)]}{g(x)g(x_0)},\end{aligned}$$

we have

$$\begin{aligned}\frac{\psi(x_0+h)-\psi(x_0)}{h} &= \frac{1}{g(x)g(x_0)}\left[\frac{f(x)-f(x_0)}{x-x_0}g(x_0)-f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right]\\ &\to \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{[g(x_0)]^2} \quad \text{as} \quad h\to 0.\end{aligned}$$

Thus, ψ is differentiable at x_0, and $\psi'(x_0)=\dfrac{g(x_0)f'(x_0)-f(x_0)g'(x_0)}{[g(x_0)]^2}$. ∎

Theorem 2.37 (Composition rule) *Suppose f is differentiable at x_0 and g is differentiable at $y_0 := f(x_0)$. Then $g\circ f$ is differentiable at x_0 and*

$$(g\circ f)'(x_0) = g'(y_0)f'(x_0).$$

Proof. Let (x_n) be a sequence in a deleted neighbourhood of x_0 which converges to x_0. We have to prove that $\lim_{n\to\infty}\dfrac{(g\circ f)(x_n)-(g\circ f)(x_0)}{x_n-x_0}$ exists and the limit is $g'(y_0)f'(x_0)$. For this, let $y_n := f(x_n)$ for $n\in\mathbb{N}$ and $y_0 = f(x_0)$. Let us look at the formal expression

$$\begin{aligned}\frac{(g\circ f)(x_n)-(g\circ f)(x_0)}{x_n-x_0} &= \frac{g(y_n)-g(y_0)}{x_n-x_0}\\ &= \frac{g(y_n)-g(y_0)}{y_n-y_0}\times\frac{f(x_n)-f(x_0)}{x_n-x_0}.\end{aligned}$$

Since $f'(x_0)$ exists, $\lim_{n\to\infty}\dfrac{f(x_n)-f(x_0)}{x_n-x_0}=f'(x_0)$. However, we will not be able write $\lim_{n\to\infty}\dfrac{g(y_n)-g(y_0)}{y_n-y_0}=g'(x_0)$, because (y_n) may not be in a deleted neighbourhood of y_0, although $y_n\to y_0$, by continuity of f at x_0. To take care of this situation, for each $n\in\mathbb{N}$, we define

$$\alpha_n = \begin{cases}\frac{g(y_n)-g(y_0)}{y_n-y_0} & \text{if} \quad y_n\neq y_0,\\ g'(y_0) & \text{if} \quad y_n = y_0.\end{cases}$$

Note that $\alpha_n\to g'(y_0)$. Hence,

$$\frac{(g\circ f)(x_n)-(g\circ f)(x_0)}{x_n-x_0} = \alpha_n\times\frac{f(x_n)-f(x_0)}{x_n-x_0}\to g'(y_0)f'(x_0)$$

showing that $(g\circ f)'(x_0)=g'(y_0)f'(x_0)$. ∎

In view of the formula in Theorem 2.37, the following result is not surprising.

Theorem 2.38 *Suppose $g \circ f$ is differentiable at x_0, g is differentiable at y_0 with $g'(y_0) \neq 0$, and f is continuous at x_0. Then f is differentiable at x_0 and*

$$f'(x_0) = \frac{(g \circ f)'(x_0)}{g'(y_0)}.$$

Proof. Let (x_n) be a sequence in a deleted neighbourhood of x_0 which converges to x_0, $y_n := f(x_n)$ for $n \in \mathbb{N}$ and $y_0 = f(x_0)$. Let (α_n) be as in the proof of Theorem 2.37. Since f is continuous at x_0, $y_n \to y_0$ so that $\alpha_n \to g'(y_0) \neq 0$ and $\alpha_n \neq 0$ for all large enough n. Then, we have

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \frac{1}{\alpha_n} \times \frac{(g \circ f)(x_n) - (g \circ f)(x_0)}{x_n - x_0} \to \frac{(g \circ f)'(x_0)}{g'(y_0)} \quad \text{as} \quad n \to \infty.$$

Thus $f'(x_0)$ exists and $f'(x_0) = \dfrac{(g \circ f)'(x_0)}{g'(y_0)}$ ∎

As a corollary to the above theorem we have the following useful formula.

Theorem 2.39 *Suppose $f : I \to J$ is bijective function between open intervals I and J. Suppose f is differentiable at a point $x_0 \in I$ and $f'(x_0) \neq 0$ and f^{-1} is continuous at x_0. Then f^{-1} is differentiable at $y_0 := f(x_0)$, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. Note that $(f \circ f^{-1})(y)) = y$ for every $y \in J$. Hence by Theorem 2.38, f^{-1} is differentiable at y_0 and $(f^{-1})'(y_0) = 1/f'(x_0)$. ∎

Remark 2.11 Recall that in Theorem 2.38 and Theorem 2.39 we assumed $g'(y_0) \neq 0$ and $f'(x_0) \neq 0$. Can we obtain atleast differentiability without the above assumptions? Note that

$$(f^{-1})'(y_0) f'(x_0) = 1.$$

Hence, Theorem 2.37 shows that the condition $f'(x_0) \neq 0$ is necessary in Theorem 2.39 for the differentiability of f^{-1} at x_0. What about the case of Theorem 2.38? In this case, f need not be differentiable at x_0 if $g'(y_0) = 0$, as the following example shows. Let

$$f(x) = |x|, \quad g(x) = x^2, \quad x \in \mathbb{R}.$$

Then $(g \circ f)(x) = x^2$ so that $g \circ f$ is differentiable at $x_0 = 0$ and g is differentiable at $y_0 := f(x_0) = 0$, but f is not differentiable at $x_0 = 0$. Note that $g'(y_0) = 0$. ♦

The derivatives of functions in the following examples, at certain points, are obtained by using the properties proved above.

Example 2.29 The following can be verified easily.

(i) For $n \in \mathbb{N}$, if $f(x) = x^n$, $x \in \mathbb{R}$, then $f'(x) = nx^{n-1}$ for $x \in \mathbb{R}$.

(ii) If $f(x) = \cos x = 1 - 2\sin^2(x/2)$, $x \in \mathbb{R}$, then $f'(x) = -\sin x$ for $x \in \mathbb{R}$.

(iii) For $x \in D := \{x \in \mathbb{R} : \cos x \neq 0\}$, let $f(x) = \tan x$. Then $f'(x) = sec^2 x$ for $x \in D$. □

Example 2.30 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

From the composition and product rules, it can be seen that f is differentiable at every $x_0 \neq 0$. Now, let us check the differentiability at $x_0 = 0$. For h in a deleted the neighbourhood of 0, we have

$$\frac{f(h) - f(0)}{h} = \frac{h\sin(1/h)}{h} = \sin(1/h).$$

Hence $f'(0)$ does not exist. □

Example 2.31 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

In this case also, is differentiable at every $x_0 \neq 0$ follows from the composition and product rules. Now, let $x_0 = 0$ and h be a deleted the neighbourhood of 0. Then

$$\frac{f(h) - f(0)}{h} = \frac{h^2\sin(1/h)}{h} = h\sin(1/h).$$

Since $0 \leq |h\sin(1/h)| \leq |h|$, $\lim_{h\to 0} h\sin(1/h)$ exists and it is equal to 0. Hence $f'(0)$ exists and $f'(0) = 0$. □

Example 2.32 Let $f(x) = x|x|$, $x \in \mathbb{R}$. Thus,

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

Note that for f is differentiable for $x \neq 0$, and $f'(x) = 2|x|$, $x \neq 0$. Now, let us check the differentiability at 0. For $x \neq 0$, we have

$$x > 0 \quad \Longrightarrow \quad \frac{f(x) - f(0)}{x} = \frac{x^2}{x} = x,$$

$$x < 0 \quad \Longrightarrow \quad \frac{f(x) - f(0)}{x} = \frac{-x^2}{x} = -x.$$

Thus, $f'_+(0) = 0$ and $f'_-(0) = 0$ so that f is differentiable at 0 and $f'(0) = 0$. Hence, $f'(x) = 2|x|$ for every $x \in \mathbb{R}$. □

Example 2.33 For $a > 0$, the function a^x is differentiable for every $x \in \mathbb{R}$ and

$$(a^x)' = a^x \ln a \qquad \forall x \in \mathbb{R}.$$

By the composition rule in Theorem 2.37,

$$(a^x)' = (e^{x \ln a})' = e^{x \ln a} \ln a = a^x \ln a.$$

□

Example 2.34 The function $\ln x$ is differentiable for every $x > 0$, and

$$(\ln x)' = \frac{1}{x}, \qquad x > 0.$$

To see this, let $f(x) = \ln x$ and $g(x) = e^x$. Then we have $g(f(x)) = x$ for every $x > 0$. Since $g \circ f$ is differentiable, g is differentiable, and $g'(y) = e^y \neq 0$ for every $y \in \mathbb{R}$, it follows by Theorem 2.37 that f is differentiable for every $x > 0$ and we have $g'(f(x))f'(x) = 1$. Thus,

$$1 = e^{\ln x}(\ln x)' = x(\ln x)'$$

so that $(\ln x)' = 1/x$. □

Example 2.35 For $a > 0$, the function $\log_a x$ is differentiable for every $x > 0$, and

$$(\log_a x)' = \frac{1}{x \ln a}, \qquad x > 0.$$

We know that

$$\log_a x = \frac{\ln x}{\ln a}.$$

Hence, $(\log_a x)' = \dfrac{1}{x \ln a}$ for every $x > 0$. □

Example 2.36 For $r \in \mathbb{R}$, let $f(x) = x^r$ for $x > 0$. Then f is differentiable for every $x > 0$ and

$$f'(x) = rx^{r-1}, \qquad x > 0.$$

By the composition rule in Theorem 2.37,

$$f'(x) = (e^{r \ln x})' = e^{r \ln x} \frac{r}{x} = \frac{x^r r}{x} = rx^{r-1}.$$

□

***Exercise* 2.18** Prove the following.

(i) The function $\ln |x|$ is differentiable for every $x \in \mathbb{R}$ with $x \neq 0$, and

$$(\ln |x|)' = \frac{1}{x}, \qquad x \neq 0.$$

(ii) For $a > 0$, the function $\log_a |x|$ is differentiable for every $x \in \mathbb{R}$ with $x \neq 0$, and

$$(\log_a |x|)' = \frac{1}{x \ln a}, \qquad x \neq 0.$$

◀

2.3.4 Maxima and minima

Recall from Theorem 2.19 that if $f : [a, b] \to \mathbb{R}$ is a continuous function, then there exists x_0, y_0 in $[a, b]$ such that

$$f(x_0) \le f(x) \le f(y_0) \qquad \forall x \in [a, b].$$

In this case, we write

$$f(x_0) = \min_{a \le x \le b} f(x) \quad \text{and} \quad f(y_0) = \max_{a \le x \le b} f(x).$$

Definition 2.19 A (real valued) function f defined on an interval I (of finite or infinite length) is said to attain

(a) **global maximum** at a point $x_1 \in I$ if $f(x) \le f(x_1)$ for all $x \in I$, and

(b) **global minimum** at a point $x_2 \in I$ if $f(x_2) \le f(x)$ for all $x \in I$.

The function f is said to attain **global extremum** at a point $x_0 \in I$ if f attains either global maximum or global minimum at x_0. □

Thus, a continuous function f defined on a closed and bounded interval I attain global maximum and global minimum at some points in I.

In Remark 2.4 we have seen that a function f defined on an interval I need not attain maximum or minimum if either I is not closed and bounded or if f is not continuous. However, maximum or minimum can attain in a subinterval. To take care of these cases, we introduce the following definition.

Definition 2.20 A (real valued) function f defined on an interval I (of finite or infinite length) is said to attain

(a) **local maximum** at a point $x_1 \in I$ if

$$f(x) \le f(x_1)$$

for all x in a deleted neighbourhood of x_1,

(b) **local minimum** at a point $x_2 \in$ if

$$f(x_2) \le f(x)$$

for all x in a deleted neighbourhood of x_2,

(c) **strict local maximum** and **strict local minimum** at x_1 and x_2, respectively, if strict inequality holds in (a) and (b), respectively.

The function f is said to attain

(d) **local extremum** at a point $x_0 \in I$ if f attains either local maximum or local minimum at x_0.

(e) **strict local extremum** at a point $x_0 \in I$ if f attains either strict local maximum or strict local minimum at x_0.

□

Remark 2.12 It is conventional to omit the adjective *local* in local maximum, local minimum and local extremum. Thus when we say a function has maximum at a point x_0, we generally mean a local maximum at x_0. Similar comments apply to minimum and extremum. ♦

***Exercise* 2.19** Suppose f is a continuous function defined on an interval I and x_0 is an interior point of I. Prove the following.

(i) If f is increasing (respectively, strictly increasing) on $(x_0 - h, x_0)$ and decreasing (respectively, strictly decreasing) on $(x_0, x_0 + h)$ for some $h > 0$, then f attains local maximum (respectively, strict local maximum) at x_0.

(ii) If "increasing" and "decreasing" in (i) above are interchanged, then in the conclusion "maximum" can be replaced by "minimum".

◀

Theorem 2.40 (A necessary condition) *Suppose f is a continuous function defined on an interval I having local extremum at a point $x_0 \in I$. If x_0 is an interior point of I (i.e., x_0 is not an end point of I) and f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Suppose f attains local maximum at x_0 which is an interior point of I. Then there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq I$ and $f(x_0) \geq f(x_0 + h)$ for all h with $|h| < \delta$. Hence, for all h with $|h| < \delta$,

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \quad \text{if} \quad h < 0,$$

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0 \quad \text{if} \quad h > 0.$$

Taking limit as $h \to 0$, we get $f'(x_0) \geq 0$ and $f'(x_0) \leq 0$ so that $f'(x_0) = 0$.

By analogous arguments, it can be shown that if f attains minimum at a point $y_0 \in (a, b)$, then $f'(y_0) = 0$. ∎

Definition 2.21 Suppose f is defined on an interval I and x_0 is an interior point of I. If $f'(x_0)$ exists and $f'(x_0) = 0$ or if $f'(x_0)$ does not exist, then x_0 is called a **critical point** of f. □

Remark 2.13 A function can have more than one maximum and minimum. For example, consider

$$f(x) = \sin(4x), \qquad [0, \pi].$$

We see that f has maximum value 1 at $\pi/8$ and $5\pi/8$, and has minimum value -1 at $3\pi/8$ and $7\pi/8$. ♦

Remark 2.14 (a) In view of Theorem 2.40, if a function f is differentiable at an interior point x_0 of an interval I and $f'(x_0) \neq 0$, then f can not have local maximum or local minimum at x_0.

(b) It is to be observed that in order to have a maximum or minimum at a point x_0, the function need not be differentiable at x_0. For example

$$f(x) = 1 - |x|, \qquad |x| \leq 1,$$

has a maximum at 0 and

$$g(x) = |x|, \qquad |x| \leq 1,$$

has a minimum at 0. Both f and g are not differentiable at 0.

(c) Also, if a function is differentiable at a point x_0 and $f'(x_0) = 0$, then it is not necessary that it has loal maximum or local minimum at x_0. For example, consider

$$f(x) = x^3, \qquad |x| < 1.$$

In this example, we have $f'(0) = 0$. Note that f has neither local maximum nor local minimum at 0. ♦

Next we give a sufficient condition of local extrema of functions. Before that we define the concept of an *increasing function* and *decreasing function*.

In Sections 2.3.6 and 2.3.7, we shall give some sufficient conditions for existence of local exrema of functions. Now, let us derive some important consequences of Theorem 2.40.

2.3.5 Rolle's theorem, mean value theorems and L'Hospital rules

Theorem 2.41 (Rolle's theorem) *Suppose f is a continuous function defined on a closed and bounded interval $[a, b]$ such that it is differentiable at every $x \in (a, b)$. If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Let $g(x) = f(x) - f(a)$. Then we have

$$g(a) = 0 = g(b) \quad \text{and} \quad g'(x) = f'(x) \quad \forall\, x \in (a, b). \tag{$*$}$$

Since g is continuous on $[a, b]$, it attains the (global) maximum and (global) minimum at some points x_1 and x_2, respectively, in $[a, b]$, i.e., there exists x_1, x_2 in $[a, b]$ such that

$$g(x_2) \leq g(x) \leq g(x_1) \qquad \forall\, x \in [a, b].$$

If $g(x_1) = g(x_2)$, then g is a constant function and hence $g'(x) = 0$ for all $x \in [a, b]$. Hence, assume that $g(x_2) < g(x_1)$. Then, either $g(x_1) \neq 0$ or $g(x_2) \neq 0$. If $g(x_1) \neq 0$, then by $(*)$, $x_1 \notin \{a, b\}$, i.e., $x_1 \in (a, b)$ so that by Theorem 2.40, $g'(x_1) = 0$ and hence, $f'(x_1) = 0$.

Similarly, if $g(x_2) \neq 0$, then we shall arrive at $f'(x_2) = 0$. ∎

***Exercise* 2.20** Show that between any two roots of the equation $e^x \cos x - 1 = 0$, there is at least one root of the equation $e^x \sin x - 1 = 0$. ◄

As a corollary to Rolle's theorem we obtain the following.

Theorem 2.42 (Mean value theorem) *Suppose f is a continuous function defined on a closed and bounded interval $[a, b]$ such that it is differentiable at every $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let

$$\varphi(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a), \qquad x \in [a, b].$$

Note that φ is continuous on $[a, b]$, differentiable in (a, b), $\varphi(a) = 0 = \varphi(b)$, and

$$\varphi'(x) := f'(x) - \frac{f(b) - f(a)}{b - a}, \qquad x \in (a, b).$$

By Rolle's theorem (Theorem 2.41), there exists $c \in (a, b)$ such that $\varphi'(c) = 0$. Thus, $f(b) - f(a) = f'(c)(b - a)$. ∎

Remark 2.15 The mean value theorem above is also called *Lagrange's mean value theorem.* ♦

Example 2.37 Let f be continuous on $[a, b]$ and differentiable at every point in (a, b). Suppose there exists $c \in \mathbb{R}$ such that

$$f'(x) = c \qquad x \in (a, b).$$

Then there exists $b \in \mathbb{R}$ such that

$$f(x) = c\,x + b \qquad \forall\, x \in [a, b].$$

In particular, $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function.

To see this consider $x_0 \in (a,b)$. Then for any $x \in [a,b]$, there exists ξ_x between x_0 and x such that

$$f(x) - f(x_0) = f'(\xi_x)(x - x_0) = c(x - x_0).$$

Hence, $f(x) = f(x_0) + c(x - x_0)$. Thus, $f(x) = c\,x + b$ with $b = f(x_0) - c\,x_0$. □

Suppose f and g are continuous functions on $[a,b]$ which are differentiable on (a,b). Suppose further that $g'(x) \neq 0$ for all $x \in (a,b)$. Then, by the mean value theorem, there exist c_1, c_2 in (a,b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c_1)}{g'(c_2)}.$$

Question is whether we can assert the existence of a single point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Answer is in affirmative as the following theorem shows.

Theorem 2.43 (Cauchy's generalized mean value theorem) *Suppose f and g are continuous functions on $[a,b]$ which are differentiable at every point in (a,b). Suppose further that $g'(x) \neq 0$ for all $x \in (a,b)$. Then, there exists $c \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

[Note that by Theorem 2.42, $g'(x) \neq 0$ for all $x \in (a,b)$ implies that $g(b) - g(a) \neq 0$.]

Proof. First note that from the assumption on g, using Mean value theorem, $g(b) \neq g(a)$. Now, let

$$\varphi(x) := f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)], \qquad x \in [a,b].$$

Note that φ is continuous on $[a,b]$, differentiable in (a,b), $\varphi(a) = 0 = \varphi(b)$, and

$$\varphi'(x) := f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x), \qquad x \in (a,b).$$

By Rolle's theorem (Theorem 2.41), there exists $c \in (a,b)$ such that $\varphi'(c) = 0$. This completes the proof. ∎

***Exercise* 2.21** Let $0 < a < b$. Show that for every $n \in \mathbb{N}$, $a < \dfrac{n[b^{n+1} - a^{n+1}]}{(n+1)[b^n - a^n]} < b$.

[Hint: take $f(x) = x^{n+1}$ and $g(x) = x^n$.] ◀

If f is defined in a closed interval $[a,b]$ and $x_0 = a$ or $x_0 = b$, then by $lim_{x \to x_0} f(x)$ we mean $lim_{x \to x_0+} f(x)$ if $x_0 = a$ and $lim_{x \to x_0^-} f(x)$ if $x_0 = b$.

Theorem 2.44 (L'Hospital's rule[2]) *Suppose functions f and g are continuous in a neighbourhood of a point x_0 and differentiable in a deleted neighbourhood of x_0. Suppose*

$$f(x_0) = 0, \quad g(x_0) = 0 \text{ and } \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \text{ exists.}$$

Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. Since $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists, there exists a deleted neighbourhood D_0 of x_0 in the domain of definition of f such that $g'(x) \neq 0$ for $x \in D_0$. By Cauchy's generalized mean value theorem (Theorem 2.43), for every $x \in D_0$, there exists ξ_x between x and x_0 such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi_x)}{g'(\xi_x)}.$$

Since $|\xi_x - x_0| < |x - x_0$ and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists, by using the limits of composition of functions, $\lim_{x\to x_0} \frac{f'(\xi_x)}{g'(\xi_x)}$ exists and it is equal to $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$. Thus, $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$. This completes the proof. ∎

The following theorem is proved by modifying the arguments in the proof of Theorem 2.44 .

Theorem 2.45 (L'Hospital's rule) *Suppose functions f and g are continuous in a neighbourhood of a point x_0 and differentiable in a deleted neighbourhood of x_0. Suppose*

$$\lim_{x\to x_0} f(x) = 0, \quad \lim_{x\to x_0} g(x) = 0 \quad \text{and} \quad \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \text{ exists.}$$

Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. Let $\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0 \end{cases}$ and $\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0 \end{cases}$. Then, the result is obtained from Theorem 2.44 by taking $\tilde{f}$ and $\tilde{g}$ in place of f and g, respectively. ∎

[2] *L'Hospital* is pronounced as *Lopital.* The rule is named after the 17th-century French mathematician Guillaume de l'Hospital, who published the rule in his book *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (i.e., Analysis of the Infinitely Small to Understand Curved Lines) (1696), the first textbook on differential calculus.

Theorem 2.46 (L'Hospital's rule) *Suppose f and g are differentiable at every point in (a, ∞) for some $a > 0$. Suppose*

$$\lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to\infty} g(x) = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \ \textit{exists.}$$

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \ \textit{exists and} \ \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Proof. Let $\tilde{f}(y) = f(1/y)$ and $\tilde{g}(y) = g(1/y)$ for $0 < y < 1/a$. We note that

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x) \iff \lim_{y\to 0} \tilde{f}(y) = 0 = \lim_{y\to 0} \tilde{g}(y).$$

Also, since

$$\tilde{f}'(y) = [f(1/y)]' = f'(1/y)(-1/y^2), \qquad \tilde{g}'(y) = [g(1/y)]' = g'(1/y)(-1/y^2),$$

we have

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} \quad \text{exists} \iff \lim_{y\to 0} \frac{\tilde{f}'(y)}{\tilde{g}'(y)} \quad \text{exists.}$$

Hence, applying Theorem 2.44 to $\tilde{f}$, $\tilde{g}$ instead of f, g, we obtain the result. ∎

Theorem 2.47 (L'Hospital's rule) *Suppose f and g are continuous functions on $[a, b]$ which are differentiable at every point in (a, b), except possibly at $x_0 \in [a, b]$. Suppose*

$$\lim_{x\to x_0} f(x) = \infty, \quad \lim_{x\to x_0} g(x) = \infty \quad \textit{and} \quad \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \ \textit{exists.}$$

Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} \ \textit{exists and} \ \lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. Let $\beta := \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$. First we consider the case of $\beta \neq 0$. In this case, since

$$\lim_{x\to x_0} f(x) = \infty = \lim_{x\to\infty} g(x) \iff \lim_{x\to x_0} (1/f(x)) = 0 = \lim_{x\to x_0} (1/g(x)),$$

the result follows from Theorem 2.45 by interchanging the roles of f and g.

To consider the general case where β is not necessarily non-zero, let x, y be distinct points in a deleted neighbourhood of x_0. Since $g'(x) \neq 0$ for x sufficiently close to x_0, in view of MVT, we can assume that $g(x) \neq g(y)$. Note that,

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(x)}{g(x)} \frac{\left[1 - \frac{f(y)}{f(x)}\right]}{\left[1 - \frac{g(y)}{g(x)}\right]} \tag{1}$$

Since $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to x_0$, the above expression is meaningful for each fixed y and x close enough to x_0, and

$$\lim_{x\to x_0}\left[1 - \frac{f(y)}{f(x)}\right] = 1 = \lim_{x\to x_0}\left[1 - \frac{g(y)}{g(x)}\right]. \tag{2}$$

Also, by GMVT, there exists $\xi_{x,y}$ lying between x and y such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi_{x,y})}{g'(\xi_{x,y})}. \tag{3}$$

From (1) and (3) above we have

$$\frac{f(x)}{g(x)} = \frac{f'(\xi_{x,y})}{g'(\xi_{x,y})}\frac{\left[1 - \frac{g(y)}{g(x)}\right]}{\left[1 - \frac{f(y)}{f(x)}\right]}. \tag{4}$$

We observe that

$$|\xi_{x,y} - x_0| \le |\xi_{x,y} - y| + |y - x_0| \le |x - y| + |y - x_0|.$$

Hence, $\xi_{x,y} \to x_0$ as $x \to x_0$ and $y \to y_0$. Hence, by using the limits of composition of functions, we obtain

$$\lim_{\alpha\to x_0}\frac{f'(\xi_{x,\alpha})}{g'(\xi_{x,\alpha})} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}. \tag{5}$$

Therefore, (2), (4), (5) imply that $\lim_{x\to x_0}\frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{\alpha\to x_0}\frac{f'(\xi_{x,\alpha})}{g'(\xi_{x,\alpha})}\frac{\left[1 - \frac{f(\alpha)}{f(x)}\right]}{\left[1 - \frac{g(\alpha)}{g(x)}\right]} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

This completes the proof. ∎

Remark 2.16 The cases

(i) $\lim_{x\to-\infty} f(x) = 0 = \lim_{x\to-\infty} g(x)$,

(ii) $\lim_{x\to x_0} f(x) = -\infty = \lim_{x\to x_0} g(x)$

can be treated analogously to the cases already discussed in the above theorems. ♦

2.3.6 Some consequences of mean value theorem

Increasing and decreasing functions

Theorem 2.48 *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then*

(i) *f is increasing iff $f'(x) \ge 0$ for all $x \in (a, b)$.*

(ii) *f is decreasing iff $f'(x) \le 0$ for all $x \in (a, b)$.*

(iii) *f is strictly increasing if $f'(x) > 0$ for all $x \in (a, b)$.*

(iv) *f is strictly decreasing if $f'(x) < 0$ for all $x \in (a, b)$.*

Proof. (i) Suppose f is increasing and $x \in (a, b)$. Then

$$\frac{f(x+h) - f(x)}{h} \ge 0$$

for all h such that $x + h \in (a, b)$. Hence $f'(x) \ge 0$.

Conversely, suppose $f'(x) \ge 0$ for all $x \in (a, b)$. Let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. Then, by mean value theorem, there exists $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1).$$

Since $f'(\xi) \ge 0$, the above equation shows that $f(x_1) \le f(x_2)$.

(ii) Follows as in the proof of (i) by reversing the inequalities.

(iii) Follows from the converse part of the proof of (i) by using $f'(\xi) > 0$.

(iv) Follows as in the converse part of the proof of (i) by using $f'(\xi) < 0$. ∎

Example 2.38 Consider the function $f(x) = x^4$ for $x \in \mathbb{R}$. Then we have $f'(x) = 4x^3$ for all $x \in \mathbb{R}$. Note that

$$f'(x) > 0 \quad \forall\, x > 0 \quad \text{and} \quad f'(x) < 0 \quad \forall\, x < 0.$$

Hence,

f is strictly increasing on $(0, \infty)$, and

f is strictly decreasing on $(-\infty, 0)$. □

A sufficient condition for local extremum point

Theorem 2.49 *Suppose f is continuous on an interval I and x_0 is an interior point of I. Further suppose that f is differentiable in a deleted nbd of x_0.*

(i) *If there exists an open interval $I_0 \subseteq I$ containing x_0 such that*

$$f'(x) > 0 \quad \forall\, x \in I_0,\ x < x_0 \quad \textit{and} \quad f'(x) < 0 \quad \forall\, x \in I_0,\ x > x_0,$$

then f has local maximum at x_0.

(ii) *If there exists an open interval $I_0 \subseteq I$ containing x_0 such that*

$$f'(x) < 0 \quad \forall\, x \in I_0,\ x < x_0 \quad \textit{and} \quad f'(x) > 0 \quad \forall\, x \in I_0,\ x > x_0,$$

then f has local minimum at x_0.

Proof. (i) Let $x \in I_0$. Then, by mean value theorem, there exists ξ_x between x_0 and x such that

$$f(x) - f(x_0) = f'(\xi_x)(x - x_0).$$

By assumption,

$$x < x_0 \Longrightarrow f'(\xi_x) > 0 \quad \text{and} \quad x > x_0 \Longrightarrow f'(\xi_x) < 0.$$

Hence, in both the cases, we have $f(x) < f(x_0)$ so that f has local maximum at x_0. Thus, (i) is proved.

Similar arguments will lead to the proof of (ii). ∎

Example 2.39 Consider

$$f(x) = x^4, \qquad g(x) = 1 - x^4, \quad |x| < 1.$$

Then $f'(x) = 4x^3$ is negative for $x < 0$ and positive for $x > 0$. Hence, by Theorem 2.49, f has local minimum at 0. Also, $g'(x) = -4x^3$ is positive for $x < 0$ and negative for $x > 0$. Hence, by Theorem 2.49, g has local maximum at 0. □

Remark 2.17 The conditions given in Theorem 2.49 cannot be dropped. For example, consider $f(x) = x^3$, $x \in \mathbb{R}$. Then $f'(x) = 3x^2 > 0$ for all $x \neq 0$. Note that f does not have extremum at 0. ♦

2.3.7 Higher derivatives and Taylor's formula

Suppose f is defined on an open interval I and $x_0 \in I$. If f is differentiable in a neighbourhood of x_0, then we can talk about the existence of *higher derivatives* of f at x_0.

Definition 2.22 Suppose f is differentiable in a neighbourhood of x_0. Then f is said to be **twice differentiable** at x_0 if the function f' is differentiable at x_0, i.e.,

$$\lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

exists, and in that case the limit is called the **second derivative** of f and it is denoted by

$$f''(x_0) \quad \text{or} \quad f^{(2)}(x_0) \quad \text{or} \quad \frac{d^2 f}{dx^2}(x_0).$$

□

Definition 2.23 For $k \in \mathbb{N}$ with $k \geq 2$, f is said to be k **times differentiable** at $x_0 \in I$ if $f^{(k-1)}$ is differentiable at x_0, and in that case

$$f^{(k)}(x_0) := [f^{(k-1)}]'(x_0)$$

is called the k^{th}**-derivative** of f at x_0, where $f^{(1)}(x)$, $f^{(2)}(x), \ldots, f^{(k-1)}(x)$ are defined iteratively as

$$f^{(j)}(x) := [f^{(j-1)}]'(x), \quad j = 1, \ldots, k-1$$

for x in a neighbourhood of x_0 with $f^{(0)}(x) = f(x)$. □

Note that $f^{(2)}(x_0)$ is the second derivative of f at x_0.

Definition 2.24 The function f is said to be **infinitely differentiable** at a point $x_0 \in I$ if for every $k \in \mathbb{N}$, f has k^{th}-derivative at x_0. □

We may observe the following:

- If f is infinitely differentiable at a point $x_0 \in I$, then for every $k \in \mathbb{N}$, f has k^{th}-derivative not only at x_0 but also at every point in some neighbourhood of x_0.

Example 2.40 For $n \in \mathbb{N}$, let $f(x) = x^n$, $x \in \mathbb{R}$. Then we know that $f^{(1)}(x) = f'(x) = nx^{n-1}$. Hence, for $k \le n$, we have

$$f^{(k)}(x) = n(n-1)\cdots(n-k+1)x^{n-k}$$

and $f^{(k)}(x) = 0$ for $k > n$. Thus, f is infinitely differentiable in $\mathbb{R}$. More generally, if f is a polynomial, then f is infinitely differentiable in $\mathbb{R}$. □

Example 2.41 Let $f(x) = \sin x$, $x \in \mathbb{R}$. Then we have

$$f^{(1)}(x) = \cos x, \quad f^{(2)}(x) = -\sin x, \quad f^{(3)}(x) = -\cos x, \quad f^{(4)}(x) = \sin x,$$

and more generally for any $k \in \mathbb{N}$,

$$f^{(2k-1)}(x) = (-1)^{k+1}\cos x, \quad f^{(2k)}(x) = (-1)^k \sin x.$$

Thus, f is infinitely differentiable in $\mathbb{R}$. □

Example 2.42 Let $f(x) = e^x$, $x \in \mathbb{R}$. We know that $f'(x) = e^x$, $x \in \mathbb{R}$. Hence, it follows that $f^{(k)}(x) = e^x$, $x \in \mathbb{R}$, for every $k \in \mathbb{N}$ so that f is infinitely differentiable in $\mathbb{R}$. □

Example 2.43 Let $f(x) = x|x|$, $x \in \mathbb{R}$. We have seen in Example 2.32 that f is differentiable at every point in $\mathbb{R}$ and $f'(x) = 2|x|$. Thus, f is infinitely differentiable at every $x \neq 0$, but differentiable only once at 0.

If $f_k(x) = x^k|x|$, $x \in \mathbb{R}$, then it can be verified that f is infinitely differentiable at every $x \neq 0$, $f^{(k)}(0)$ exists, but $f^{(k+1)}(0)$ does not exist. □

Taylor's formula

Our next attempt is to express a function f which is $n+1$ times differentiable in a neighbourhood of a point x_0 as

$$f(x) = f(x_0) + \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)^{n+1}, \qquad (*)$$

where ξ_x is a point lying between x_0 and x. The above formula $(*)$ is called **Taylor's formula**. Before establishing $(*)$, let us look at a situation when f is a polynomial.

Suppose $f(x)$ is a polynomial of degree $n \in \mathbb{N}$ and $x_0 \in \mathbb{R}$. Since $f(x) - f(x_0)$ vanishes at $x = x_0$, we can write

$$f(x) = f(x_0) + (x - x_0)f_1(x),$$

where $f_1(x)$ is a polynomial of degree $n-1$. By the same argument, if $n > 1$, then f_1 can be written as

$$f_1(x) = f_1(x_0) + (x - x_0)f_2(x),$$

where $f_2(x)$ is a polynomial of degree $n-2$. Thus,

$$f(x) = f(x_0) + f_1(x_0)(x - x_0) + (x - x_0)f_2(x).$$

Continuing this, there are polynomials $f_1(x), f_2(x), \ldots, f_{n-2}(x), f_{n-1}(x), f_n(x)$ of degree $n-1, n-2, \ldots, 2, 1, 0$, respectively, such that

$$f(x) = f(x_0) + f_1(x_0)(x - x_0) + f_2(x)(x - x_0)^2 + \cdots + f_n(x_0)(x - x_0)^n.$$

Note that

$$f^{(1)}(x_0) = f_1(x_0), \quad f^{(2)}(x_0) = 2!f_2(x_0), \ldots, f^{(n)}(x_0) = n!f_n(x_0),$$

so that

$$f(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Now, suppose that f is a function which is $n+1$ times differentiable in a neighbourhood of x_0 for some $k \in \mathbb{N}$. If we write,

$$P(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

then we can write

$$f(x) = P(x) + R(x)$$

where $R(x) := f(x) - P(x)$ is $n+1$ times differentiable and $R(x_0) = 0$. We may also observe that

$$R^{(k)}(x_0) = 0 \quad \text{for} \quad k = 1, \ldots, n.$$

Taylor's formula gives a specific expression for $R(x)$ in terms of the $(n+1)^{\text{th}}$ derivative of f at a point ξ lying between x_0 and x.

Theorem 2.50 (Taylor's formula) *Suppose f is defined and has derivatives $f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n+1)}(x)$ for x in a neighbourhood I_0 of a point x_0. Then, for every $x \in I_0$, there exists ξ_x between x and x_0 such that*

$$f(x) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)^{n+1}.$$

Proof. Let $x \in I$ with $x \neq x_0$, and let

$$P_n(y) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}(y - x_0)^j, \qquad y \in I.$$

Then $P_n(y)$ is a polynomial of degree n, $P_n(x_0) = f(x_0)$ and

$$P_n^{(j)}(x_0) = f^{(j)}(x_0), \qquad j \in \{1, \ldots, n\}.$$

Now, let

$$g(y) = f(y) - P_n(y) - \varphi(x)(y - x_0)^{n+1}, \qquad y \in I,$$

where

$$\varphi(x) := \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}.$$

Note that, by this choice of $\varphi(x)$, we have $g(x_0) = 0$ and $g(x) = 0$. Also, we have

$$g^{(1)}(x_0) = 0, \quad g^{(2)}(x_0) = 0, \quad \ldots, \quad g^{(n)}(x_0) = 0.$$

Since $g(x_0) = 0 = g(x)$, by Rolle's theorem, there exists x_1 between x_0 and x such that $g'(x_1) = 0$. Since $g'(x_0) = 0 = g'(x_1)$, again by Rolle's theorem, there exists x_2 between x_0 and x_1 such that $g''(x_2) = 0$. Continuing this, there exists $\xi_x := x_{n+1}$ between x_0 and x_n such that $g^{(n+1)}(\xi_x) = 0$. But,

$$g^{(n+1)}(y) = f^{(n+1)}(y) - P_n^{(n+1)}(y) - \varphi(x)(n+1)! = f^{(n+1)}(y) - \varphi(x)(n+1)!.$$

Thus, using the fact that $g^{(n+1)}(\xi_x) = 0$, we have

$$\varphi(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}.$$

Thus,

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)^{n+1},$$

and the proof is complete. ∎

Proof using Cauchy's GMVT. Let $R_n(x) = f(x) - P_n(x)$, where

$$P_n(x) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j.$$

Since $f^{(k)}(x_0) = P_n^{(k)}(x_0)$ for $k = 0, 1, \ldots, n$, we have

$$R_n(x_0) = 0, \quad R_n'(x_0) = 0, \quad \ldots, R_n^{(n)}(x_0) = 0.$$

Let $\psi(x) = (x - x_0)^{n+1}$. Now, let $x \in I_0$, $x \neq x_0$. Since $\psi(x_0) = 0$ and $\psi'(x) \neq 0$, by Cauchy's generalized mean value theorem (GMVT), there exists x_1 between x_0 and x such that

$$\frac{R_n(x)}{\psi(x)} = \frac{R_n(x) - R_n(x_0)}{\psi(x) - \psi(x_0)} = \frac{R_n'(x_1)}{\psi'(x_1)}.$$

Again, since $R_n'(x_0) = 0 = \psi'(x_0)$ and $\psi''(x) \neq 0$, by GMVT, there exists x_2 between x_0 and x_1 such that

$$\frac{R_n'(x_1)}{\psi'(x_1)} = \frac{R_n'(x_1) - R_n'(x_0)}{\psi'(x) - \psi'(x_0)} = \frac{R_n''(x_2)}{\psi'(x_2)}.$$

Continuing this, at the $(n+1)^{\text{th}}$ stage, there exists x_n between x_0 and x_n such that

$$\frac{R_n^{(n)}(x_n)}{\psi^{(n)}(x_n)} = \frac{R_n^{(n)}(x_n) - R_n^{(n)}(x_0)}{\psi^{(n)}(x_n) - \psi^{(n)}(x_0)} = \frac{R_n^{(n+1)}(x_{n+1})}{\psi^{(n+1)}(x_{n+1})} = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}.$$

Thus,

$$R_n(x) = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}\psi(x) = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}(x - x_0)^{n+1}.$$

This completes the proof. ▮

Remark 2.18 The first and second proofs given above for Theorem 2.50 are adapted from the books [3] and [2], respectively. In the next Chapter we shall give another, rather *simpler* proof for this. ◆

Definition 2.25 In the Taylor's formula (Theorem 2.50), the polynomial

$$P_n(x) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j$$

is called the **Taylor's polynomial** of f of degree n around x_0, and the term

$$R_n(x) := \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)^{n+1}$$

is called the **remainder term** in the formula. □

We observe that if f is infinitely differentiable and if

$$|R_n(x)| \to 0 \quad \text{as} \quad n \to \infty$$

for every $x \in I$, then

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad x \in I.$$

Definition 2.26 If f is infinitely differentiable in a neighbourhood of x_0 and if it can be represented as a series

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad x \in I$$

for all x in a neighbourhood of x_0, then such a series is called the **Taylor series** of f around the point x_0. If $x_0 = 0$, the corresponding Taylor series is called the **Maclaurin series** of f. □

Observe that if $f^{(n+1)}$ is bounded in a neighbourhood of x_0, i.e., there exists $M_n > 0$ such that say $|f^{(n+1)}(x)| \leq M_n$ for all x in that neighbourhood, then

$$|f(x) - P_n(x)| \leq \frac{M_n|x - x_0|^{n+1}}{(n+1)!}.$$

In particular, if f is infinitely differentiable, and if there exists $M > 0$, independent of n such that $|f^{(n+1)}(x)| \leq M$ for all x in a neighbourhood I_0 of x_0, then

$$|f(x) - P_n(x)| \leq \frac{M|x - x_0|^{n+1}}{(n+1)!} \to 0$$

so that f has the Taylor series expansion

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

for all $x \in I_0$.

Remark 2.19 A natural question that one may ask is:

> Does every infinitely differentiable function in a neighbourhood of x_0 has a Taylor's series expansion?

Unfortunately, the answer is negative. For example, if we define

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then it can be seen that $f(0) = 0$ and $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Thus, f does not have the Taylor's series expansion around the point 0. ♦

Example 2.44 Let $f(x) = e^x$ for $x \in \mathbb{R}$. Then we know that $f^{(k)}(x) = e^x$ so that for any $x_0, x \in \mathbb{R}$,

$$R_n(x) := \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)^{n+1} = \frac{e^{\xi_x}}{(n+1)!}(x - x_0)^{n+1}.$$

Since $e^{\xi_x} \leq \psi(x) := \max\{e^{x_0}, e^x\}$, we have

$$|R_n(x)| \leq \psi(x)\frac{|x - x_0|^{n+1}}{(n+1)!} \to 0.$$

Hence, f has the Taylor series expansion

$$e^x = e^{x_0}\left[1 + \sum_{n=1}^{\infty}\frac{(x - x_0)^n}{n!}\right].$$

for every $x, x_0 \in \mathbb{R}$. Observe that the function which represents the series within the bracket is nothing but e^{x-x_0}. □

Example 2.45 Using Taylor's formula, we shall show that

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \forall\, x \in \mathbb{R}.$$

For this, let $f(x) = \sin x$ and $x_0 = 0$. Since f is infinitely differentiable, and

$$f^{2j}(0) = 0, \qquad f^{2j-1}(0) = (-1)^j \qquad \forall\, j \in \mathbb{N},$$

we have

$$\begin{aligned} f(x) &= f(x_0) + \sum_{j=1}^{2n+1}\frac{f^{(j)}(0)}{j!}x^j + \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!}x^{2n+2} \\ &= f(x_0) + \sum_{j=0}^{n}\frac{f^{(2j+1)}(0)}{(2j+1)!}x^{2j+1} + \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!}x^{2n+2} \\ &= f(x_0) + \sum_{j=0}^{n}\frac{(-1)^j}{(2j+1)!}x^{2j+1} + \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!}x^{2n+2} \end{aligned}$$

Also, since $|\sin x| \leq 1$, we have

$$\left|\frac{f^{(2n+2)}(\xi_x)x^{2n+2}}{(2n+2)!}\right| \leq \frac{|x|^{2n+2}}{(2n+2)!} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\left|f(x) - \left[f(x_0) + \sum_{j=0}^{n}\frac{(-1)^j}{(2j+1)!}x^{2j+1}\right]\right| \to 0 \quad \text{as} \quad n \to \infty$$

and hence, $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \forall\, x \in \mathbb{R}.$ □

***Exercise* 2.22** Suppose f is infinitely differentiable in an open interval I and $x_0 \in I$. Further, suppose that there exists $M > 0$ such that

$$|f^{(k)}(x)| \le M \qquad \forall x \in I, \quad \forall k \in \mathbb{N} \cup \{0\}.$$

Then show that

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad x \in I.$$

◀

***Exercise* 2.23** Using Taylor's formula, prove the following:

(i) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$.

(ii) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for all x with $|x| < 1$.

(iii) $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for all $x \in \mathbb{R}$.

(iv) From (iii), deduce then Madhava-Gregory series for $\pi/4$, i.e., $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

◀

Another sufficient condition for extremum points

Theorem 2.51 *Suppose f is defined on an interval I and x_0 is an interior point of I. Suppose that x_0 is a critical point of f, i.e., $f'(x_0) = 0$, and f has continuous continuous second derivative in a neighbourhood of x_0. Then we have the following:*

(i) If $f''(x_0) < 0$, then f has local maximum at x_0.

(ii) If $f''(x_0) > 0$, then f has local minimum at x_0.

Proof. By Taylor's theorem, there exists an open interval I_0 containing x_0 such that for every $x \in I_0$, there exists ξ_x between x_0 and x such that

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(\xi_x)}{2}(x - x_0)^2 = \frac{f''(\xi_x)}{2}(x - x_0)^2. \qquad (*)$$

(i) Suppose $f''(x_0) < 0$. Since f'' is continuous in a nbd of x_0, there exists an open interval I_1 containing x_0 such that for all $x \in I_1$,

$$f''(x) \le \frac{f''(x_0)}{2}.$$

In particular, from $(*)$, we obtain

$$f(x) - f(x_0) = \frac{f''(\xi_x)}{2}(x - x_0)^2 < 0 \qquad \forall x \in I_1.$$

Thus, f has a maximum at x_0.

(ii) Suppose $f''(x_0) > 0$. Then, we obtain reverse of the inequalities in the proof of (i), and arrive the conclusion that f has a minimum at x_0. ∎

Remark 2.20 The conditions given in Theorem 2.51 are only sufficient conditions. There are functions f for which none of the conditions (i) and (ii) are satisfied at a point x_0, still f can have local extremum at x_0. For example, consider

$$f(x) = x^4, \qquad g(x) = 1 - x^4, \quad |x| < 1.$$

Then $f'(0) = 0 = g'(0)$, f has local minimum at 0 and g has local maximum at 0. But, $f''(0) = 0 = g''(0)$. ♦

Remark 2.21 How to identify critical points and extreme points of a function?

1. Suppose f is defined on an open interval I.
   (a) Find those points at which either f is not differentiable or f' vanish. These points are the critical points of f.
   (b) Suppose $f'(x_0) = 0$.
      i. If $f'(x)$ has the same sign for x on both side of x_0, then f does not have an extremum at x_0. Otherwise,
      ii. use the test for maximum or minimum as given in Theorem 2.49.
2. Suppose f is continuous on $[a, b]$ and differentiable on (a, b).
   (a) f can have maximum or minimum only the at the end points of $[a, b]$ or at those points in (a, b) at which f' vanishes.
   (b) Use the tests as in Theorem 2.49 or Theorem 2.51.

♦

2.3.8 Determination of shapes of a curves

We shall use conditions on derivatives of a function to find out certain nature of the curve determined by a function. First we spell out what is meant by a curve determined by a function.

Definition 2.27 Let f be a continuous function defined on an interval I. Then the graph of f, i.e.,

$$G_f := \{(x, f(x) : x \in I\},$$

is called a **curve determined by** f. □

A curve determined by a function $f : I \to \mathbb{R}$ is often written as

$$y = f(x), \quad x \in I.$$

Definition 2.28 Let f be a continuous function defined on an interval I. Then the curve determined by f is said to be

1. **convex upwards** or **concave downwards** if f is differentiable at all interior points of I and the tangent line at each point $x \in I$ lies above the curve,
2. **concave upwards convex downwards** if f is differentiable at all interior points of I and the tangent line at each point $x \in I$ lies below the curve.

□

Thus, if f is defined on an interval I and differentiable at all interior points of I, then the curve determined by f is

- convex upwards if and only if for any interior point x_0 of I,

$$x \in I \setminus \{x_0\},\ y = f(x_0) + f'(x_0)(x - x_0) \quad \Longrightarrow \quad f(x) < y,$$

- convex downwards if and only if for any interior point x_0 of I,

$$x \in I \setminus \{x_0\},\ y = f(x_0) + f'(x_0)(x - x_0) \quad \Longrightarrow \quad f(x) > y.$$

Theorem 2.52 *Let f be a continuous function defined on an interval I. Suppose f has second derivative at all interior points of I. Then the curve determined by f is*

(i) *convex upwards if $f''(x) < 0$ for all interior points x in I, and*

(ii) *convex downwards if $f''(x) > 0$ for all interior points x in I.*

Proof. Suppose $f''(x) < 0$ for all interior points x in I. Let x_0 be any point in the interior of I. We have to show that

$$x \in I,\ y = f(x_0) + f'(x_0)(x - x_0) \quad \Longrightarrow \quad f(x) < y.$$

So let $x \in I$ and $y = f(x_0) + f'(x_0)(x - x_0)$. By Taylor's theorem, there exists c_x between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c_x)}{2}(x - x_0)^2$$

so that, using the fact that $f''(c_x) < 0$,

$$f(x) = y + \frac{f''(c_x)}{2}(x - x_0)^2 < y.$$

Hence, G_f is convex upwards, proving (i). Proof of (ii) follows analogously. ∎

Example 2.46 (i) Let $f(x) = x^2$ and $g(x) = 1 - x^2$ for $x \in \mathbb{R}$. Then G_f is convex downwards and G_g is convex upwards.

(ii) Let $f(x) = e^x$, $x \in \mathbb{R}$. Note that $f''(x) > 0$ for all $x \in \mathbb{R}$. Hence, by the Theorem 2.52, $y = e^x$ is convex downwards on $\mathbb{R}$.

(iii) Let $f(x) = x^3$, $x \in \mathbb{R}$. Note that $f''(x) = 6x$ so that, by the Theorem 2.52, the curve $y = x^3$ is convex upwards for $x < 0$ and convex downwards for $x > 0$. □

Definition 2.29 A point (x_0, y_0) on the the curve determined by a function f is said to be a **point of inflection** of the curve if in a neighbourhood of x_0, the curve is convex upward on one side of x_0 and convex downward on other side of x_0. □

Example 2.47 In view of the conclusions in Example 2.46 (iii), the point $(0, 0)$ on the curve $y = x^3$ is a point of inflexion. □

Theorem 2.53 *Suppose f has second derivative in a deleted neighbourhood of a point x_0. Then the point $(x_0, f(x_0))$ is a point of inflection of the curve G_f if f'' has constant but different signs on each side of x_0, and at at the point x_0, either $f''(x_0)$ does not exist or $f''(x_0) = 0$.*

Proof. This a consequence of Theorem 2.52. ∎

Theorem 2.54 *Suppose f has second derivative in a neighbourhood I_0 of a point x_0. If $(x_0, f(x_0))$ is a point of infection of the curve G_f and if f'' is continuous at x_0, then $f''(x_0) = 0$.*

Proof. Suppose $(x_0, f(x_0))$ is a point of infection of the curve G_f and f'' is continuous at x_0. Without loss of generality, assume that G_f is convex upward for $x \in I_0$, $x < x_0$ and it is convex downward for $x \in I_0$, $x > x_0$. Thus,

$$x \in I_0,\ x < x_0 \quad \Longrightarrow \quad f(x) < f(x_0) + f'(x_0)(x - x_0), \tag{1}$$

$$x \in I_0,\ x > x_0 \quad \Longrightarrow \quad f(x) > f(x_0) + f'(x_0)(x - x_0). \tag{2}$$

So, let $x \in I_0$. By Taylor's theorem, there exists c_x between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c_x)}{2}(x - x_0)^2.$$

Now, (1) implies that $f''(c_x) < 0$ so that by letting $x \to x_0$, we have $f''(x_0) \le 0$. Also, (2) implies that $f''(c_x) > 0$ so that by letting $x \to x_0$, we have $f''(x_0) \ge 0$. Thus, $f''(x_0) = 0$. ∎

2.4 Additional exercises

2.4.1 Limit

1. Using the definition of limit, show that $\lim_{x \to 3} \frac{x}{4x - 9} = 1$.

2. Show that the function f defined by $f(x) = \begin{cases} x, & \text{if } x < 1, \\ 1 + x, & \text{if } x \ge 1 \end{cases}$ does not have the limit as $x \to 1$.

3. Let f be defined by $f(x) = \begin{cases} 3 - x, & \text{if } x > 1, \\ 1, & \text{if } x = 1, \\ 2x, & \text{if } x < 1. \end{cases}$

   Find $\lim_{x \to 1} f(x)$. Is it $f(1)$?

4. Let f be defined on a deleted neighbourhood D_0 of a point x_0 and $\lim_{x \to x_0} f(x) = b$. If $b \neq 0$, then show that there exists $\delta > 0$ such that $f(x) \neq 0$ for every $x \in (x_0 - \delta, x_0 + \delta) \cap D_0$.

5. Let f be defined by $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$ Show that

   (i) $\lim_{x \to 0} f(x)$ does not exist, and

   (ii) $\lim_{x \to 0} x f(x) = 0$.

6. Suppose $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to \infty} g(x) = b$. Show that $\lim_{x \to \infty} g(f(x)) = b$.

7. Let $f : (0, \infty) \to \mathbb{R}$ be such that $\lim_{x \to 0} f(x) = b$. Show that $\lim_{x \to \infty} f(x^{-1}) = b$.

8. Verify the following.

   (a) If $\lim_{x \to \infty} f(x) = b$ and $\lim_{x \to \infty} g(x) = c$, then

   $$\lim_{x \to \infty} [f(x) + g(x)] = b + c, \qquad \lim_{x \to \infty} f(x)g(x) = bc.$$

(b) If If $\lim_{x\to\infty} f(x) = b$, $\lim_{x\to\infty} g(x) = c$ and $c \neq 0$, then there exists $M_0 > 0$ such that $g(x) \neq 0$ for all $x > M_0$ and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{b}{c}.$$

9. State and prove sequential characterization for

$$\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to a} f(x) = -\infty, \quad \lim_{x\to +\infty} f(x) = \infty,$$

$$\lim_{x\to +\infty} f(x) = -\infty, \quad \lim_{x\to -\infty} f(x) = \infty, \quad \lim_{x\to -\infty} f(x) = -\infty.$$

2.4.2 Continuity

1. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. If $c \in (a, b)$ is such that $f(c) > 0$, and if $0 < \beta < f(c)$, then show that there exists $\delta > 0$ such that $f(x) > \beta$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the relation $f(x + y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$. If f is continuous at 0, then show that f is continuous at every $x \in \mathbb{R}$, and in that case $f(x) = xf(1)$ for every $x \in \mathbb{R}$.

3. There does not exist a continuous function f from $[0, 1]$ onto $\mathbb{R}$ – Why?

4. Find a continuous function f from $(0, 1)$ onto $\mathbb{R}$.

5. Suppose $f : [a, b] \to [a, b]$ is continuous. Show that there exists $c \in [a, b]$ such that $f(c) = c$.

6. There exists $x \in \mathbb{R}$ such that $17x^{19} - 19x^{17} - 1 = 0$ – Why?

7. If $p(x)$ is a polynomial of odd degree, then there exists at least one $\xi \in \mathbb{R}$ such that $p(\xi) = 0$.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $f(x) \to 0$ as $|x| \to \infty$. Prove that f attains either a maximum or a minimum.

9. Suppose $f : [a, b] \to \mathbb{R}$ is continuous such that for every $x \in [a, b]$, there exists a $y \in [a, b]$ such that $|f(y)| \leq \frac{|f(x)|}{2}$. Show that there exists $\xi \in [a, b]$ such that $f(\xi) = 0$.

10. Suppose $f : [a, b] \to [a, b]$ is continuous such that there $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$ for all $x, y \in [a, b]$. Show that there exists $\xi \in [a, b]$ such that $f(\xi) = \xi$.

11. Write details of the proof of Corollary 2.20.

12. Prove the followiing.

(a) Let $f : (a,b) \to \mathbb{R}$ be a continuous function. If $f(x) \to c$ as $x \to a$ and $f(x) \to d$ as $x \to b$, where c, d, then for every $y \in (c,d)$, there exists $x \in (a,b)$ such that $f(x) = y$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $f(x) \to c$ as $x \to -\infty$ and $f(x) \to d$ as $x \to \infty$, where c, d, then for every $y \in (c,d)$, there exists $x \in \mathbb{R}$ such that $f(x) = y$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $f(x) \to c$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$, where c, d, then for every $y \in (c,\infty)$, there exists $x \in \mathbb{R}$ such that $f(x) = y$.

13. From Problem 12, deduce that for every $y \in (0,\infty)$, there exists $x \in \mathbb{R}$ such that $e^x = y$.

14. Prove that if f is strictly monotonic on an interval I, then f is injective on I.

15. Let f be a continuous function defined on an interval I. Show that if f is injective, then it is strictly monotonic on I [Hint: Use Intermediate Value Theorem].

16. Let f be a continuous function defined on an interval I. Show that if f is injective, then its inverse from its range is continuous.

2.4.3 Differentiation

1. Prove that the function $f(x) = |x|$, $x \in \mathbb{R}$ is not differentiable at 0.

2. Consider a polynomial $p(x) = a_0 + a_1x^2 + \ldots + a_nx^n$ with real coefficients $a_0, a_1, \ldots, a_n$ such that $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_n}{n+1} = 0$. Show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = 0$.

   [Note that the conclusion need not hold if the condition imposed on the coefficients is dropped. To see this, consider $p(x) = 1 + x^2$.]

3. Let I and J be open intervals and $f : I \to J$ be bijective and differentiable at every $x_0 \in I$. If $f'(x_0) \neq 0$, then show that the inverse function $f^{-1} : J \to I$ is also differentiable at x_0 and and $(f^{-1})'(x_0) = 1/f'(x_0)$.

4. Using Taylor's theorem, show that
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n.$$

5. Show that there does not exist a function $f : [0,1] \to \mathbb{R}$ which is differentiable on $(0,1)$ such that $f'(x) = \begin{cases} 0, & \text{if } 0 < x < 1/2, \\ 1, & \text{if } 1/2 \le x < 1. \end{cases}$

   [Hint: Use Example 2.37 in the interval $[0,1/2]$ and $[1/2,1]$ taking $x_0 = 1/2$, and show that the resulting function f is not differentiable at $x_0 = 1/2$.]