Uniform Continuity

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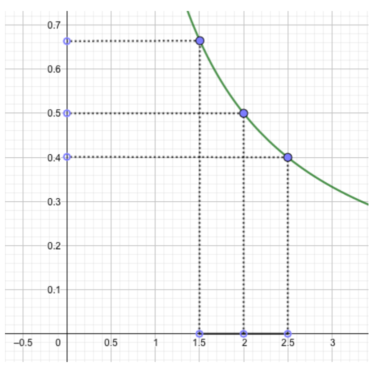
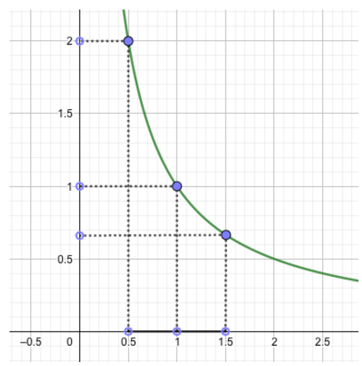
Suppose that $f : I \to \mathbb{R}$ is continuous on I, where I is any subset* of $\mathbb{R}$. What do we mean by saying that 'continuous on I'? We mean that, for every $\varepsilon > 0$, and for each $a \in I$, there exists $\delta > 0$ such that for every $x \in I$ satisfying $|x-a| < \delta$, it must hold that $|f(x)-f(a)| < \varepsilon$. Intuitively, it says that for each $a \in I$, $f(x)$ can be made arbitrarily close to $f(a)$ by making x sufficiently close to a. Now, you might ask, how close should be x to a, in order to have $|f(x) - f(a)| < \varepsilon$? We understand that δ should depend on ε. Should δ also depend on a? Let us consider some examples.

1. Consider $f : (0, \infty) \to \mathbb{R}$ defined as $f(x) = 1/x$. Take $\varepsilon = 0.2$. For $a = 2$, we note that taking $\delta = 0.5$ suffices, because when $x \in (2-0.5, 2+0.5)$, i.e. $x \in (1.5, 2.5)$, the diagram below shows that $|f(x) - f(2)| < 0.7 - 0.5 = \varepsilon$.





Now, consider $a = 1$. Does the same δ work? i.e. if $|x - 1| < 0.5$, is it necessary that $|f(x)-f(1)| < 0.2$? If you look at the second diagram above, you will notice that the same δ does not work, because $f(x)$ changes at a higher rate in $(0.5, 1)$ than in $(1.5, 2)$. For instance, we have $x = 0.6$ that lies in $(1 - \delta, 1 + \delta)$, but $|f(0.6) - f(1)| = \frac{1}{0.6} - 1 = \frac{2}{3} > \varepsilon$. Thus, we see that the choice of δ not only depends on ε, but also depends on the choice of a. Although we considered only one specific value of δ, convince yourself that any δ that works for some value of a, will not work for some other value of a. We shall prove this when we return to this example once again.

*We assume here that I does not have any 'isolated point'. For example, if $I = [1, 2] \cup \{3\}$, then 3 will be an isolated point of I. If I is the union of some intervals, then I does not have any isolated point.

2. Consider $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + \sin x$. Recall that $|\sin x - \sin y| \le |x-y|$ holds for all $x, y \in \mathbb{R}$. So, we can write $|f(x) - f(y)| = |2(x-y) + (\sin x - \sin y)| \le 2|x-y| + |\sin x - \sin y| \le 3|x-y|$ for every $x, y \in \mathbb{R}$. Hence, for every $\varepsilon > 0$ and any $a \in \mathbb{R}$, we can choose $\delta = \varepsilon/3$, so that whenever $|x-a| < \delta$, we have $|f(x) - f(a)| \le 3|x-a| < 3\delta = \varepsilon$. Thus, for this function, the same δ works for every $a \in \mathbb{R}$.

Motivated from the above two examples, we are ready to see a definition of Uniform Continuity. We say that f is uniformly continuous on I, if the choice of δ depends only on ε, i.e. the same δ works for every $a \in I$. In symbols,

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall a \in I, \text{ if } x \in I \text{ satisfies } |x-a| < \delta \text{ then } |f(x) - f(a)| < \varepsilon. \quad (1)$$

Observe that it is same as saying that

$$\forall \varepsilon > 0, \exists \delta > 0 : \text{ if } a, b \in I \text{ satisfies } |a-b| < \delta \text{ then } |f(a) - f(b)| < \varepsilon. \quad (2)$$

Remark. The definition of f being continuous on I states

$$\forall \varepsilon > 0, \forall a \in I, \exists \delta > 0 : \text{if } x \in I \text{ satisfies } |x-a| < \delta \text{ then } |f(x) - f(a)| < \varepsilon. \quad (3)$$

If we compare it with (1), we see that the only difference is that the position of '$\forall a \in I$' and '$\exists \delta > 0$' are interchanged. Why does this make a difference? Here is an example. Let C be the set of cities in India. Consider the following statements:
(i) $\forall c \in C, \exists n \in \mathbb{N} : n$ is the pincode of city c.
(ii) $\exists n \in \mathbb{N} : \forall c \in C, n$ is the pincode of city c.
Do you see how the meaning gets completely changed? In general, we can not interchange $\forall$ and $\exists$, without changing the meaning of the sentence.

Definition. Suppose $f : I \to \mathbb{R}$ is a function, where $I \subseteq \mathbb{R}$. We say that f is uniformly continuous on I if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $a, b \in I$ satisfying $|a-b| < \delta$, it holds that $|f(a) - f(b)| < \varepsilon$.

Intuitively, f is said to be uniformly continuous if it is possible to guarantee that $f(x)$ and $f(y)$ will be as close to each other as we please by requiring only that x and y are sufficiently close to each other.

Example. $f(x) = 2x + \sin x$ is uniformly continuous on $\mathbb{R}$. This follows directly from our discussion above. In fact, we showed that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| < \varepsilon/3$.

Example. $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$. To prove this, we need to show that for some $\varepsilon > 0$, no $\delta > 0$ works, i.e. for every choice of δ, there exists a pair x, y such that $|x - y| < \delta$, but $|f(x) - f(y)| \geq \varepsilon$. To prove this, fix any $\varepsilon > 0$ and take any $\delta > 0$. There exists $M > 0$ large enough so that $M > \varepsilon$ as well as $M > 1/2\delta$. Then, $x = \frac{1}{M}$ and $y = \frac{1}{2M}$ are satisfying $|x - y| = \frac{1}{2M} < \delta$, but $|f(x) - f(y)| = M > \varepsilon$.

Problem 1. Draw the graphs of $f(x) = x$, $g(x) = x^2$, $h(x) = \sqrt{x}$ and tell which of these are uniformly continuous on $[0, 1]$. Also find out which of these are uniformly continuous on $[0, \infty)$.

Problem 2. Suppose $f : I \to \mathbb{R}$ is a function that satisfies $|f(x) - f(y)| \leq k|x - y|$ for every $x, y \in I$, where $k > 0$ is a fixed real number. Show that f is uniformly continuous on I.

Problem 3. Suppose that $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous on $[0, \infty)$. Is it necessary that there exists k such that $|f(x) - f(y)| \leq k|x - y|$ for every $x, y \geq 0$?

Problem 4. Suppose $f, g : I \to \mathbb{R}$ are both uniformly continuous on I. If both of them are bounded on I then show that their product fg is uniformly continuous on I. Without the condition of boundedness, show that it is not necessary that fg is uniformly continuous on I.

Problem 5. Suppose that $g : I \to J$ and $f : J \to \mathbb{R}$ such that both f, g are uniformly continuous on their respective domains. Show that their composition $f \circ g$ is uniformly continuous on I. (Recall that, $(f \circ g)(x)$ means $f(g(x))$.)

Theorem. Suppose that $f : I \to \mathbb{R}$ is continuous on $I \subseteq \mathbb{R}$. If I is closed and bounded, then f must be uniformly continuous on I.

Proof. Let, if possible, f be not uniformly continuous. Then, there exists some $\varepsilon_0 > 0$ such that for any $\delta > 0$, there exists $a, b \in I$ satisfying $|a - b| < \delta$, but $|f(a) - f(b)| \geq \varepsilon_0$. Hence, for every $n \in \mathbb{N}$, we can take $\delta = 1/n$ to get a, b such that

$|a-b| < 1/n$ and $|f(a)-f(b)| > \varepsilon_0$. Call these a, b to be a_n, b_n. Thus, we get hold of two sequences a_n and b_n inside I such that for every $n \geq 1$,

$$|a_n - b_n| < 1/n \quad \text{and} \quad |f(a_n) - f(b_n)| \geq \varepsilon_0$$

Now, since I is a bounded, we can apply Bolzano-Weierstrass theorem to conclude that there exists two subsequences a_{n_k} and b_{n_k} (of a_n and b_n respectively) which are convergent and the limit belongs to I (since I is closed). Say, $a_{n_k} \to \ell_1$ and $b_{n_k} \to \ell_2$. Since $|a_{n_k} - b_{n_k}| < 1/n_k$ and $1 \leq n_1 < n_2 < \cdots$, we let $k \to \infty$ to conclude that $|\ell_1 - \ell_2| \leq 0$, i.e. $\ell_1 = \ell_2 = \ell$ (say). Now, since f is continuous on I and $\ell \in I$, we have

$$\lim_{k\to\infty} f(a_{n_k}) = f(\ell) = \lim_{k\to\infty} f(b_{n_k}).$$

Therefore, $f(a_{n_k}) - f(b_{n_k}) \to 0$ as $k \to \infty$. But this contradicts the fact that $|f(a_{n_k}) - f(b_{n_k})| > \varepsilon_0$ for every $k \geq 1$. □

Problem 6. Consider $f(x) = 1/x$ for $x > 0$. For every $a > 0$, show that f is uniformly continuous on $(a, 1)$. Is f uniformly continuous on $(0, 1)$?

Problem 7. Let $f : A \to \mathbb{R}$ be uniformly continuous on A. If $B \subseteq A$, show that f must be uniformly continuous on B as well.

Problem 8. Suppose that $f : (a, b) \to \mathbb{R}$ is a continuous function. If the one-sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist, then show that f must be uniformly continuous on (a, b).

Problem 9. Show that the function $f(x) = \cos\left(\frac{1}{x}\right)$ is *not* uniformly continuous on $(0, 1]$.

Problem 10. Is the function $f(x) = \frac{\sin x}{x}$ uniformly continuous on $(0, \infty)$?

Problem 11. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and satisfies the following conditions: $f(0) = 0$ and $f(x + y) \leq f(x) + f(y)$ for every $x, y \in \mathbb{R}$. Prove that f must be uniformly continuous on $\mathbb{R}$.

Problem 12. Suppose that f is continuous on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k f\left(\frac{k}{n}\right) = 0.$$

Solutions of the Problems

1. Looking at their graphs, we can tell that each of these functions are uniformly continuous on $[0,1]$. If we consider the domain to be $[0,\infty)$, then $g(x)=x^2$ does not remain uniformly continuous; the other two functions are indeed uniformly continuous on $[0,\infty)$.

   [If you want a more rigorous solution: $f(x)=x$ is obviously uniformly continuous on any domain. For $g(x)=x^2$, note that when $x\in[0,1]$, we have $|x^2-y^2| = |x+y||x-y| \le 2|x-y|$. So, g is uniformly continuous on $[0,1]$. However, g is not uniformly continuous on $[0,\infty)$. Because $|(x+\delta)^2-x^2| = 2x\delta+\delta^2 > 2x\delta$, which can be made arbitrarily large, even for very tiny δ. The last one, $h(x)=\sqrt{x}$ is uniformly continuous on $[0,1]$ as well as on $[0,\infty)$. To prove this, we note that $|\sqrt{x}-\sqrt{y}| \le \sqrt{|x-y|}$ for all $x,y\ge 0$. (To prove this, assume $x\ge y$.)]

2. Note that $\delta=\varepsilon/k$ works uniformly.

3. No. Take $f(x)=\sqrt{x}, x\ge 0$.

4. Use the ineq. $|f(x)g(x)-f(y)g(y)| \le |f(x)|\cdot|g(x)-g(y)|+|g(y)|\cdot|f(x)-f(y)|$. To see why we require f,g to be bounded, take $f(x)=g(x)=x$ and $I=[0,\infty)$.

5. $|x-y|<\delta' \implies |g(x)-g(y)|<\delta \implies |f(g(x))-f(g(y))|<\varepsilon$.

6. Fix any $a>0$. If $x,y\in(a,1)$ then $\left|\dfrac{1}{x}-\dfrac{1}{y}\right| = \dfrac{|x-y|}{xy} \le \dfrac{1}{a^2}|x-y|$, which implies that $f(x)=1/x$ is uniformly continuous on $(a,1)$, for any $a>0$. However, f is not uniformly continuous on $(0,1)$. We have already discussed it above.

7. Proof follows from the definition.

8. We define a new function g on $[a,b]$ by setting $g(x)=f(x)$ for $x\in(a,b)$, $g(a)=\lim_{x\to a^+} f(x)$, and $g(b)=\lim_{x\to b^-} f(x)$. Since g is continuous on $[a,b]$, it must be uniformly continuous on this interval. Hence g must be uniformly continuous on (a,b), and in this interval g is same as f.

9. Use the fact that $f\Big(\dfrac{1}{2n\pi}\Big) - f\Big(\dfrac{1}{2n\pi+\pi/2}\Big) = 1$ for every $n\ge 1$.

10. Write $f(x) = \sin x/x$. Fix any $\varepsilon > 0$. Since $\lim_{x\to\infty} f(x) = 0$, there exists $M > 0$ such that $|f(x)| < \varepsilon$ for every $x \geq M$. Hence, $|f(x)-f(y)| < \varepsilon$ holds for any $x, y \geq M$. Next, since we have $\lim_{x\to 0+} f(x) = 1$, we can say that f is uniformly continuous on $(0, M]$. This implies that there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ holds for every $x, y \in (0, M]$ satisfying $|x-y| < \delta$. Finally, let $x \in (0, M]$, and $y > M$, such that $|x-y| < \delta$. Then, $|x-M| < |x-y| < \delta \implies |f(x)-f(M)| < \varepsilon$. We also have $|f(M) - f(y)| < \varepsilon$. Hence, $|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| < 2\varepsilon$. Thus, we have shown that for any $x, y > 0$, $|x - y| < \delta \implies |f(x) - f(y)| < 2\varepsilon$. Since ε is arbitrary, we are done.

11. Since f is continuous at 0 and $f(0) = 0$, we can say that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x| < \delta \implies |f(x)| < \varepsilon$. Now, for $|t| < \delta$, and any $x \in \mathbb{R}$,

$$f(x+t) - f(x) \leq f(t) < \varepsilon \text{ and } f(x) - f(x+t) \leq f(-t) < \varepsilon.$$

12. Fix any $\varepsilon > 0$. Since f is uniformly continuous on $[0, 1]$, there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. And there exists n_0 such that for every $n \geq n_0$, we have $1/n < \delta$. Then, we can write $|f(\frac{k}{n}) - f(\frac{k-1}{n})| < \varepsilon$ for each $k = 1, 2, \ldots, n$. This holds for every $n \geq n_0$. We shall consider even and odd values of n separately. Observe that,

$$\left|\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k f\left(\frac{k}{2n}\right)\right| \leq \frac{1}{2n}\sum_{k=1}^{n}\left|f\left(\frac{2k-1}{2n}\right) - f\left(\frac{2k}{2n}\right)\right| \leq \frac{1}{2n}\sum_{k=1}^{n}\varepsilon = \frac{\varepsilon}{2}.$$

Similarly,
$$\begin{aligned}
&\left|\frac{1}{2n+1}\sum_{k=1}^{2n+1}(-1)^k f\left(\frac{k}{2n+1}\right)\right| \\
&\leq \frac{1}{2n+1}\sum_{k=1}^{n}\left|f\left(\frac{2k-1}{2n+1}\right) - f\left(\frac{2k}{2n+1}\right)\right| + \frac{|f(1)|}{2n+1} \\
&\leq \frac{n\varepsilon}{2n+1} + \frac{|f(1)|}{2n+1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \quad \text{(for sufficiently large } n\text{)}
\end{aligned}$$

Thus, we conclude that for all sufficiently large n, $\left|\frac{1}{n}\sum_{k=1}^{n}(-1)^k f\left(\frac{k}{n}\right)\right| < \varepsilon$ holds. Since ε is arbitrary, we are through.