Ramanujan School of Mathematics

Class Test on Calculus

November 2019

Maximum marks: 50 Time: 2 hours.

Attempt all the questions. Answers without proper explanations will fetch zero. Show all your rough work – partial solutions may be rewarded. You can use any theorem/result without proving it again; but you have to state it properly.

1. Let $f : [0, 2] \to \mathbb{R}$ be a differentiable function such that $f(0) > 0$, $f(1) = 1$ and $f'(1) > 1$. Show that there must exist $c \in (0, 1)$ such that $f(c) = c$.

2. For $x > 0$, prove that $1 + x/2 - x^2/8 < \sqrt{1 + x} < 1 + x/2$.

3. Determine, with proof, whether the following statements are true or false: (If true then provide a proof, else provide a counter-example.) $(5 \times 3 = 15)$
   (a) If $\lim_{n\to\infty} n(f(1/n) - f(0)) = 0$ then $f'(0)$ must exist and equal to 0.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim_{x\to 0} f'(x) = 0$ then $f'(0) = 0$.
   (c) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is bounded, then f must be uniformly continuous on $\mathbb{R}$.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and let $f(x) \neq 0$ for every $x \in (a, b)$. Show that there exists $\theta \in (a, b)$ such that
$$\frac{f'(\theta)}{f(\theta)} = \frac{1}{a - \theta} + \frac{1}{b - \theta}.$$

5. You are making cylindrical containers of a given volume, say V. Suppose that the top and bottom surfaces are made of a material that is k times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find the ratio of the height to the radius of the cylinder that minimizes the cost of making the containers.

Do not cheat to yourself. All the best!

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