The Number e

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Abstract

The number e peeps out in several places in Calculus. We come across more than one definitions and several properties of this number; and more often than not we are left confused by thinking how there can be so many different approaches to the same number. In this article, we seek connections between those various results involving the number e.

1 Introduction

The number e is seen in several places in Calculus, for example

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{1}$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \tag{2}$$

$$\lim_{x\to 0}\frac{e^x - 1}{x} = 1 \tag{3}$$

$$\frac{d}{dx}(e^x) = e^x \tag{4}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{5}$$

Among these, $(1), (2), (5)$ are directly defining e. And $(3), (4)$ are properties of e; but they can also be used to define e :

$$a = e \text{ is the unique number for which } \lim_{x\to 0}\frac{a^x-1}{x} = 1 \text{ and } \frac{d}{dx}(a^x) = a^x. \tag{6}$$

Now, its natural to ask, how are the different definitions give the same e ? And how the above equations relate to one another? In this article, we are going to settle down these questions.

2 Defining e

I think the first time one encounters with e is while learning logarithms, where textbooks introduce e as an 'irrational' number defined by

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots . \tag{7}$$

And then they say log to the base e is called the 'natural' logarithm. Now, for obvious reasons we are left confused with questions like "why irrational?" , "why natural?". We shall resolve these questions later. Before that, let us begin by defining e using that series. But for that, we should first ensure that the above series 'converges'.

Define $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ for $n \geq 1$. We know, the series in (7) converges if and only if the sequence $\{s_n\}_{n\geq 1}$ converges. Observe that the sequence s_n is increasing. And using the inequality $n! = 2 \cdot 3 \cdot 4 \cdot \ldots n \geq 2^{n-1}$ (for every $n \geq 2$) we get

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots = 3.$$

Thus, the sequence s_n is increasing and bounded, hence converges to some real number. We name that real number to be e :

$$e := \lim_{n\to\infty} s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

Next, we shall see another famous definition of e, with a historical story. Jacob Bernoulli discovered this constant in 1683 by studying a question about compound interest:

An account starts with \$1.00 and pays 100% interest per year. If the interest is credited once, at the end of the year, the value of the account at year-end will be \$2.00. What happens if the interest is computed and credited more frequently during the year? If the interest is credited twice in the year, the interest rate for each 6 months will be 50%, so the initial \$1 is multiplied by 1.5 twice, yielding $\$1.00 \times 1.5^2 = \$2.25$ at the end of the year. Compounding quarterly yields $\$1.00 \times (1 + 1/4)^4 = \$2.4414...$, and compounding monthly yields $\$1.00 \times (1 + 1/12)^{12} = \$2.613035...$ If there are n compounding intervals, the interest for each interval will be $100/n$ % and the value at the end of the year will be $\$1.00(1 + 1/n)^n$.

Bernoulli noticed that if we make n larger and larger (and hence the compounding intervals get smaller and smaller), then this sequence approaches a limit - with continuous compounding, the account value will reach \$2.7182818... . This gives an alternate definition of e: we write $e = \lim_{n\to\infty}(1 + 1/n)^n$.

We shall first establish that this limit exists or not. But there is a new problem: even if we show that the limit exists, we can not redefine e; rather we have to show the two definitions are equivalent. So we need to show that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots . \tag{8}$$

Let us write $t_n = (1+\frac{1}{n})^n$ for $n \geq 1$. The Binomial Theorem gives

$$\Big(1+\frac{1}{n}\Big)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!}\ \frac{n(n-1)\dots(n-k+1)}{n^k}$$
$$= \sum_{k=0}^{n} \frac{1}{k!}\ \Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\Big(1-\frac{k-1}{n}\Big)$$

This gives us a motivation for (8) to be true: when n gets larger and larger, the last sum gets closer and closer to the sum $\sum_{k=1}^{n} 1/k!$. That is,

$$\Big(1+\frac{1}{n}\Big)^n = \sum_{k=0}^{n} \frac{1}{k!}\ \Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\Big(1-\frac{k-1}{n}\Big) \approx \sum_{k=0}^{n} \frac{1}{k!} \ \text{ for large } \ n.$$

This was just a motivation. Let us do everything rigorously now : Note that,

$$t_n = \Big(1+\frac{1}{n}\Big)^n = \sum_{k=0}^{n} \frac{1}{k!}\ \Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\Big(1-\frac{k-1}{n}\Big) \leq \sum_{k=0}^{n} \frac{1}{k!} = s_n$$

Therefore, for every $n \geq 1$ we have $t_n \leq s_n$. On the other hand, for $n \geq m$,

$$\sum_{k=0}^{n} \frac{1}{k!}\ \Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\Big(1-\frac{k-1}{n}\Big) \geq \sum_{k=0}^{m} \frac{1}{k!}\ \Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\Big(1-\frac{k-1}{n}\Big) \dots (*)$$
$$\geq \sum_{k=0}^{m} \frac{1}{k!}\ \Big(1-\frac{1}{m}\Big)\Big(1-\frac{2}{m}\Big)\Big(1-\frac{k-1}{m}\Big).$$

This gives $t_n \geq t_m$ whenever $n \geq m$. Thus, the sequence t_n is increasing and $t_n \leq s_n < 3$ (we showed $s_n < 3$ earlier), hence t_n converges. Say $t = \lim\limits_{n\to\infty} t_n$. Now, $t_n \leq s_n$ tells us that $t \leq e$. And letting $n \to \infty$, $(*)$ tells us that $t \geq s_m$. Then, letting $m \to \infty$, we get $t \geq e$. Combining these two, we obtain $t = e$, as required.

Next, let us establish why e is irrational. Let, if possible, e be rational, say $e = p/q$ where p, q are positive integers. So, we have

$$\frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots.$$

Multiplying both sides of last equation with $q!$, we get

$$p \cdot (q-1)! = q!\Big(1 + + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{q!}\Big) + q!\Big(\frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \dots\Big)$$

Thus, $x = q!\Big(\frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \dots\Big) = \frac{1}{(q+1)} + \frac{1}{(q+2)(q+1)} + \dots$ must be an integer. But,

$$\frac{1}{(q+1)} + \frac{1}{(q+2)(q+1)} + \dots \leq \frac{1}{(q+1)} + \frac{1}{(q+1)^2} + \dots = \frac{1}{1-1/(1+q)} - 1 = \frac{1}{q}$$

which shows that $0 < x < 1$, so x can't be an integer. This contradiction proves that e must be irrational.

3 The Big Trap

Let me present it in the form of a conversation:

Teacher: How to show $\dfrac{d}{dx}(e^x) = e^x$?

Student 1: We have $\dfrac{d}{dx}(e^x) = \lim\limits_{h\to 0} \dfrac{e^{x+h} - e^x}{h} = e^x \lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = e^x$.

Teacher: Tell me how to prove $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

Now student 1 has no answer. Then, another student says:

Student 2: Why not show $\dfrac{d}{dt}\log t = \dfrac{1}{t}$ directly from definition! We have

$$\frac{d}{dx}(\log x) = \lim_{h\to 0} \frac{\log(x+h) - \log x}{h} = \lim_{h\to 0} \frac{\log(1+h/x)}{h} = \frac{1}{x} \lim_{u\to 0} \frac{\log(1+u)}{u} = \frac{1}{x}.$$

Teacher: Okay, then tell how to show $\lim\limits_{u\to 0} \dfrac{\log(1+u)}{u} = 1$?

Student 2: We substitute $1+u = e^z$. As $u \to 0$, $e^z = 1+u \to 1$ so $z \to 0$. Hence

$$\lim_{u\to 0} \frac{\log(1+u)}{u} = \lim_{z\to 0} \frac{z}{e^z - 1} = 1.$$

Teacher: So both of you are using $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$. But how to prove this one ?

After a little silence, another student says

Student 3: We know, $e^h = 1 + h + \dfrac{h^2}{2!} + \dfrac{h^3}{3!} + \dots$, hence

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \left(\frac{h}{2!} + \frac{h^2}{3!} + \dots\right) = 0.$$

Teacher: And how do you establish that series ?

Student 3: That's the Taylor Series for e^x. We have, for $f(x) = e^x$,

$$e^h = f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + \cdots = 1 + h + \frac{h^2}{2!} + \cdots .$$

Teacher: And how do you get $f'(0) = 1$?

Student 3: Because $f'(x) = e^x$.

Student 1: But to prove $f'(x) = e^x$, we needed that limit; and you are proving that limit using $f'(x) = e^x$!!!

Teacher: Correct. This is the trap!

4 Escaping from the trap

To escape the trap, we take a somewhat reverse route. We define

$$L(x) = \int_1^x \frac{1}{t} dt \text{ for } x > 0.$$

We shall deduce that it is same as our familiar $\log_e x$, i.e. $L(x) = \log_e x$ or, $e^{L(x)} = x$ for all $x > 0$. (Note that we can not directly evaluate the integral, because we don't know how to differentiate $\log_e x$, for now.)

Note: we are using the convention that $\displaystyle\int_b^a f = -\int_a^b f$ if $a < b$.

4.1 Showing $L(x) = \log_e x$

First we show that, $L(xy) = L(x) + L(y)$ for all $x, y > 0$.

$$L(xy) - L(y) = \int_y^{xy} \frac{dt}{t} = \int_1^x \frac{du}{u} \text{ (substituting } t = yu) .$$

In the intermediate step, the substitution is justified, because the functions $f(t) = 1/t$ and $g(s) = ys$ are nice (continuous, differentiable, bijective in required domains). Thus, for every $x, y > 0$, we have $L(xy) = L(x) + L(y)$.

Using this, we get $L(x^2) = 2L(x)$, which yields $L(x^3) = L(x) + L(x^2) = 3L(x)$ and so on. Inductively, we get $L(x^n) = nL(x)$ for all $x > 0$ and for every natural number n.

Next, observe that, using the substitution $t = 1/z$,

$$L\Big(\frac{1}{x}\Big) = \int_1^{1/x} \frac{dt}{t} = \int_1^x z\frac{-dz}{z^2} = \int_1^x -\frac{dz}{z} = -L(x)$$

Combining this with $L(x^n) = nL(x)$ and using that $L(x^0) = L(1) = 0$, we obtain

$$L(x^n) = nL(x) \text{ for all } x > 0, \text{ for all integer } n.$$

Next, for any rational number $\alpha = p/q$ where p, q are integers, $q \neq 0$, note that

$$qL(x^\alpha) = L(x^{q\alpha}) = L(x^p) = pL(x) \Rightarrow L(x^{p/q}) = \frac{p}{q}L(x).$$

Hence, $L(x^\alpha) = \alpha L(x)$ for all rational number α. Next, take any real number α. There exists a sequence of rational numbers $(a_n)_{n\geq 1}$ with $\lim a_n = \alpha$. Now, from continuity of exponential the function[1],

$$L(x^\alpha) = L(\lim_{n\to\infty} x^{a_n}) = \lim_{n\to\infty} L(x^{a_n}) = \lim_{n\to\infty} a_n L(x) = \alpha L(x).$$

[1]Continuity of exponential functions (a^x) is not something that can be proved, because it is what defines the exponentiation. How to define 2^3 is clear to us : it is simply 2 times 2 times 2. But how does one define $2^{\sqrt{2}}$? Following is the story of how to define a^x:

First we define a^n for any real number a and natural number n, by saying a^n is $a \times a \times \ldots (n$ times). Then we define a^n for all integer n, by saying $a^{-n} = 1/a^n$ for any natural number n and $a^0 = 1$; but this time we have to exclude $a = 0$. Because 0^{-1} is undefined. Next, we define a^n for all rational n by saying $a^{p/q}$ is the q-th root of a^p. But this time we have to exclude all the negative a's because $(-1)^{1/2}$ is not a real number. (It can shown that for any $a > 0$ and natural number q, $a^{1/q}$ exists and is unique).

Finally, we define a^x for all $a > 0$ and x real by taking limit of a^{x_n} where x_n is any sequence of rational numbers tending to x. For instant, $2^{\sqrt{2}}$ is the limit of 2^{x_n} where (x_n) is any sequence that tends to $\sqrt{2}$. So, continuity is what defines exponentiation, for arbitrary real exponents.

Thus, we have shown that $L(x^\alpha) = \alpha L(x)$ for all $x > 0$ and for all $\alpha \in \mathbb{R}$.

Next, we shall show that $L(e) = 1$. We know, the sequence $x_n = (1 + 1/n)^n$ tends to e as $n \to \infty$. And note that $L(x)$ is a continuous function, by Fundamental Theorem of Calculus. So continuity of L tells us that $L(x_n)$ tends to $L(e)$ as $n \to \infty$.

$$L(e) = \lim_{n\to\infty} L\Big(1 + \frac{1}{n}\Big)^n = \lim_{n\to\infty} nL\Big(1 + \frac{1}{n}\Big) = \lim_{n\to\infty} n \int_1^{1+1/n} \frac{dt}{t}$$

Now, for $t \in [1, 1 + 1/n]$, we have $\dfrac{1}{1 + 1/n} \leq \dfrac{1}{t} \leq 1$. Hence,

$$\frac{1/n}{1 + 1/n} = \int_1^{1+1/n} \frac{dt}{1 + 1/n} \leq \int_1^{1+1/n} \frac{dt}{t} \leq \int_1^{1+1/n} dt = \frac{1}{n}$$

which gives $\dfrac{1}{1 + 1/n} \leq n \displaystyle\int_1^{1+1/n} \frac{dt}{t} \leq 1$. Now, letting $n \to \infty$, we get

$$L(e) = \lim_{n\to\infty} n \int_1^{1+1/n} \frac{dt}{t} = 1.$$

Finally, observe that, for every $x > 0$,

$$L(x) = L(e^{\log_e x}) = (\log_e x)L(e) = \log_e x \text{ (as } L(e) = 1).$$

4.2 Derivatives of $\log x, e^x$ and the limit $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h}$

First we shall show that $\frac{d}{dx}(\log x) = 1/x$ for all $x > 0$. The integrand $f(t) = 1/t$ is continuous in $(1, \infty)$. Hence, Fundamental Theorem of Calculus tells us that $L(x) = \int_1^x f(t)dt$ is differentiable in $(1, \infty)$ with $L'(x) = f(x) = 1/x$ for all $x > 1$. For $x < 1$, we can use $L(x) = -L(1/x)$ to arrive at

$$L'(x) = \frac{d}{dx}\left(-L\left(\frac{1}{x}\right)\right) \overset{chain\ rule}{=} -\frac{1}{1/x} \cdot \frac{d}{dx}\left(\frac{1}{x}\right) = (-x)\frac{-1}{x^2} = \frac{1}{x}$$

Now we are left with only $L'(1)$. For this, we rewrite $L(x)$ as

$$L(x) = \int_{0.5}^x dt/t - \int_{0.5}^1 dt/t = \int_{0.5}^x dt/t + \text{some constant}.$$

Then, continuity of $1/t$ in $[0.5, \infty)$ ensures L is differentiable in $(0.5, \infty)$, with $L'(x) = f(x) = 1/x$ for all $x > 0.5$. In particular, we have $L'(1) = 1$.

Next, we shall find the derivative of e^x. Observe that, the function $h : \mathbb{R} \to (0, \infty)$, $h(x) = e^x$ is a bijective function with inverse $h^{-1}(x) = \log x$ (because $y = e^x$ implies $x = \log y$). To put it in another way, e^x is the inverse of the differentiable function $\log x$ and the derivate of $\log x$ (which is $1/x$) is non-zero for $x \in (0, \infty)$. Hence e^x is differentiable.

Differentiating both sides of $x = \log(e^x)$ w.r.t. x (using chain-rule for the RHS),

$$1 = \frac{d}{dx}(\log(e^x)) = \frac{1}{e^x} \cdot \frac{d}{dx}(e^x) \implies \frac{d}{dx}(e^x) = e^x.$$

Finally, we have

$$e^x = \frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} \implies \lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

4.3 Alternate ways to define e

Using $a^x = e^{x \log a}$ we obtain

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to 0} \frac{e^{x\log a} - 1}{x} = \log a \lim_{x\to 0} \frac{e^{x\log a} - 1}{x \log a} = \log a,$$

and
$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\log a}) = (e^{x\log a}) \log a = a^x \log a.$$

Hence we can say that

$$a = e \text{ is the unique number for which } \lim_{x\to 0} \frac{a^x - 1}{x} = 1 \text{ and } \frac{d}{dx}(a^x) = a^x \text{ holds.}$$

Also, notice that $\int_1^a \frac{1}{t} dt = \int_1^e \frac{1}{t} dt + \int_e^a \frac{1}{t} dt = 1 + \int_e^a \frac{1}{t} dt$ tells us $\int_1^a \frac{1}{t} dt = 1$ if and only if $a = e$.

At this point, I can attempt give you a possible answer of the question: Why do we call $\log_e x$ to be the *natural* logarithm of x? One can guess that the derivative of a^x (w.r.t. x) is proportional to itself[1]. Now, think about guessing that proportionality constant. Well, we might do one thing - set some number as *standard* and measure the proportionality constant with respect to that. Hence we seek whether there is a number a for which that proportionality constant becomes 1. Quite dramatically, it turns out that this number is none other than e. Not only that, it is the unique choice for a which makes both the *limit* $\lim_{x\to 0}(a^x - 1)/x$ and the *area* $\int_1^a \frac{1}{x} dx$ to be 1. Due to these reasons, in calculus (which comprises of *limit, derivatives* and *area*) we set this number as the *standard* one. In other words, the number e is the most *natural* constant in calculus.

4.4 The Series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Once we know that the derivative of e^x is itself, the above series is just the Taylor series of e^x. The error after truncating the series at the n-th term, is of the form $e^{\xi_n} x^n/n!$ where $0 < |\xi_n| < |x|$. We have $0 < e^{\xi_n} \leq 1 + e^x$, so this part of the error is bounded. Hence the conclusion will follow once we show $\lim_{n\to\infty} x^n/n! = 0$. To prove this, consider the series $\sum_{n=1}^{\infty} x^n/n!$. By ratio-test, this series converges (for any real number x), hence the n-th term of the series must tend to 0 as $n \to \infty$. This completes the proof that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

[1] Watch 3blue1brown's video at https://youtu.be/m2MIpDrF7Es

However, the proof above gives us no motivation how a seemingly different series relates with e. Let me give that motivation now. Observe that,

$$\frac{d}{dx}\Big(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}\Big)=\Big(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}\Big).$$

So, the derivative of $\Big(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\Big)$ is almost like itself, with the last term being erased! Now, what if we continue adding all such terms (making a power series)? We expect that

$$\frac{d}{dx}\Big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\dots\Big)=\Big(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\dots\Big) \quad (\star)$$

Since there is no last term, the derivative of the series is just itself. And this is a property the function e^x enjoys! Of course, e^x is not the only one having this property: for every $c \in \mathbb{R}$, the function ce^x has the same property. But, after all, its natural to ask, is that series related to e^x ? And the answer turns out to be 'yes', because the series happens to be nothing but the Taylor series expansion for e^x. (We proved that the series converges to e^x, for any real number x.)

The reader might get a bit disgusted if I give no explanation for differentiating the series in $(\star)$ term-by-term. Note that the in $(\star)$ is a power series (around 0), which converges for any real number x, as seen by ratio-test. So the radius of convergence of the power series is $R=+\infty$ and we can differentiate a power series (around 0) term-by-term within $(-R,R)$. Thus, the term-by-term differentiation in $(\star)$ is justified.