Stirling's Approximation

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It is difficult to compute $n!$ even for moderately large n, like $n = 100$. However, there is a nice approximation for $n!$, named Stirling's Approximation, that states

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}e^{-n}n^{n+1/2}} = 1. \qquad (*)$$

To see how good this approximation is, you can compare $f(n) = \sqrt{2\pi}e^{-n}n^{n+1/2}$ with $n!$ for $n = 5, 10, 20$ (and so on). Note that $(*)$ tells us that $n!/f(n)$ goes to 1. So, if you consider $n! - f(n)$, you might get disappointed to see that this difference does not get smaller and smaller. Instead, we should consider the relative error,

$$\frac{n! - f(n)}{n!}$$

which goes to 0 as n increases, as seen from $(*)$. We shall see a proof of $(*)$ below, divided into a number of smaller problems.

Problem 1. Suppose $\{x_n\}_{n\geq 1}$ is a sequence such that $\sum_{n=1}^{\infty} |x_{n+1} - x_n|$ converges. Then $\lim_{n\to\infty} x_n$ exists.

Solution. Since $\sum_{n=1}^{\infty} |x_{n+1} - x_n|$ converges, so $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$ must converge as well. This means that $\lim_{N\to\infty} \sum_{n=1}^{N-1}(x_{n+1} - x_n)$ exists, which is same as saying $\lim_{N\to\infty} x_N$ exists. □

Problem 2. For $0 < x < 1$, we have $\left|\log(1 + x) - x + \frac{x^2}{2}\right| \leq \frac{x^3}{3}$.

Solution. Fix any $x \in (0, 1)$. By Taylor's theorem, there exists $c \in (0, x)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(c).$$

Applying it for $f(x) = \log(1 + x)$, we get

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3!}\frac{2}{(1 + c)^3}$$

where $0 < c < x < 1$. The conclusion follows immediately from this equation. □

Problem 3. Show that $\lim_{n\to\infty} \frac{n!}{e^{-n}n^{n+1/2}}$ exists. We shall denote this limit by C.

Solution. For $n \geq 1$, let us define

$$x_n = \log\left(\frac{n!}{e^{-n}n^{n+1/2}}\right) = \log n! + n - \left(n + \frac{1}{2}\right)\log n.$$

First observe that,

$$\begin{aligned} x_{n+1} - x_n &= \log(n+1) - \frac{1}{2}\log\left(1 + \frac{1}{n}\right) + 1 - (n+1)\log(n+1) + n\log n \\ &= -\left(n + \frac{1}{2}\right)\log\left(1 + \frac{1}{n}\right) + 1. \end{aligned}$$

Using problem 2, we have $\left|\log\left(1 + \frac{1}{n}\right) - \frac{1}{n} + \frac{1}{2n^2}\right| < \frac{1}{3n^3}$. Multiplying this inequality by n and $\frac{1}{2}$ respectively, and then using triangle inequality, we get

$$\left|\left(n + \frac{1}{2}\right)\log\left(1 + \frac{1}{n}\right) - 1\right| \leq \left|n\log\left(1 + \frac{1}{n}\right) - 1 + \frac{1}{2n}\right| + \left|\frac{1}{2}\log\left(1 + \frac{1}{n}\right) - \frac{1}{2n}\right|$$
$$\leq \left(n + \frac{1}{2}\right)\left|\log\left(1 + \frac{1}{n}\right) - \frac{1}{n} + \frac{1}{2n^2}\right| + \frac{1}{4n^2} \leq \left(n + \frac{1}{2}\right)\frac{1}{3n^3} + \frac{1}{4n^2} < \frac{1}{n^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} |x_{n+1} - x_n| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Hence, we can use problem 1 to conclude that $\lim_{n\to\infty} x_n$ exists, say ℓ. Finally, since $x \mapsto e^x$ is continuous, the required limit exists, and $\lim_{n\to\infty} e^{x_n} = e^{\ell} =: C$. □

Problem 4. Let $I_n = \int_0^{\pi/2} (\sin x)^n \, dx$ for $n \geq 0$. Show that,

1. $I_n \geq I_{n+1}$ for every $n \geq 0$.
2. $I_n = \frac{n-1}{n} I_{n-2}$ for every $n \geq 2$.
3. Find a formula for I_{2n+1} and I_{2n}.
4. $\lim_{n\to\infty} \frac{I_{n+1}}{I_n} = 1$.

Solution. The first two are left for the reader (the second one can be shown using integration by parts). For the third one, we use the last part repetitively.

$$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1} = \frac{2n}{(2n+1)} \frac{(2n-2)}{(2n-1)} I_{2n-3} = \cdots = \frac{2n}{(2n+1)} \frac{(2n-2)}{(2n-1)} \cdots \frac{2}{3} I_1$$

and $I_1 = 1$. Similarly,

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2} = \cdots = \frac{(2n-1)}{2n}\frac{(2n-3)}{(2n-2)}\cdots\frac{1}{2}I_0 = \frac{(2n-1)}{2n}\frac{(2n-3)}{(2n-2)}\cdots\frac{1}{2}\frac{\pi}{2}.$$

For the last one, we observe that for each $n \geq 1$,

$$I_n \leq I_{n-1} = \frac{n+1}{n} I_{n+1} \implies \frac{n}{n+1} \leq \frac{I_{n+1}}{I_n} \leq 1.$$

Hence, Sandwich theorem applies and gives us the desired limit. □

Problem 5. Show that, $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \frac{C^2}{2\pi}$ and hence conclude that $C = \sqrt{2\pi}$.

Proof. From the last problem, we have

$$\frac{I_{2n+1}}{I_{2n}} = \left(\frac{2n(2n-2)\cdots 2}{(2n-1)(2n-3)\cdots 1}\right)^2 \frac{2}{\pi(2n+1)} = \left(\frac{(2^n n!)^2}{(2n)!}\right)^2 \frac{2}{\pi(2n+1)}.$$

Denote $g(n) = e^{-n}n^{n+1/2}$. Problem 3 tells us that $\lim_{n\to\infty} \frac{n!}{g(n)} = C$. Therefore,

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{(2n)!^2}\frac{2}{\pi(2n+1)} = \lim_{n\to\infty} \frac{2^{4n}g(n)^4}{g(2n)^2}\frac{2C^2}{\pi(2n+1)}$$

Observe that, $\frac{2^{4n}g(n)^4}{g(2n)^2} = 2^{4n}\frac{e^{-4n}n^{4n+2}}{e^{-4n}(2n)^{4n+1}} = \frac{n}{2}$. Hence, the last limit becomes

$$\lim_{n\to\infty} \frac{2^{4n}g(n)^4}{g(2n)^2}\frac{2C^2}{\pi(2n+1)} = \lim_{n\to\infty} \frac{nC^2}{\pi(2n+1)} = \frac{C^2}{2\pi}.$$

Now, the last problem tells us $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$. Therefore, $C^2 = 2\pi \implies C = \sqrt{2\pi}$ (because $C \geq 0$). This completes our proof. □

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Any application? Applications of Stirling's formula can be found in different parts of Probability theory. For example, it is used in the proof of the de Moivre-Laplace theorem, which states that the normal distribution may be used as an approximation to the binomial distribution under certain conditions. It is also used in study of Random Walks.

See also: What is the purpose of Stirling's approximation to a factorial? (asked in math.stackexchange.com).