Example 4.33. The function

$$f(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable, but not continuously differentiable, at 0 and $f'(0) = 1/2 > 0$. However, f is not increasing in any neighborhood of 0 since

$$f'(x) = \frac{1}{2} - \cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$$

is continuous for $x \neq 0$ and takes negative values in any neighborhood of 0, so f is strictly decreasing near those points.

4.5. Taylor's theorem

If $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and $f' : (a, b) \to \mathbb{R}$ is differentiable, then we define the second derivative $f'' : (a, b) \to \mathbb{R}$ of f as the derivative of f'. We define higher-order derivatives similarly. If f has derivatives $f^{(n)} : (a, b) \to \mathbb{R}$ of all orders $n \in \mathbb{N}$, then we say that f is infinitely differentiable on (a, b).

Taylor's theorem gives an approximation for an $(n + 1)$-times differentiable function in terms of its Taylor polynomial of degree n.

Definition 4.34. Let $f : (a, b) \to \mathbb{R}$ and suppose that f has n derivatives $f', f'', \dots f^{(n)} : (a, b) \to \mathbb{R}$ on (a, b). The Taylor polynomial of degree n of f at $a < c < b$ is

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{1}{2!} f''(c)(x - c)^2 + \dots + \frac{1}{n!} f^{(n)}(c)(x - c)^n.$$

Equivalently,

$$P_n(x) = \sum_{k=0}^{n} a_k (x - c)^k, \qquad a_k = \frac{1}{k!} f^{(k)}(c).$$

We call a_k the kth Taylor coefficient of f at c. The computation of the Taylor polynomials in the following examples are left as an exercise.

Example 4.35. If $P(x)$ is a polynomial of degree n, then $P_n(x) = P(x)$.

Example 4.36. The Taylor polynomial of degree n of e^x at $x = 0$ is

$$P_n(x) = 1 + x + \frac{1}{2!} x^2 \cdots + \frac{1}{n!} x^n.$$

Example 4.37. The Taylor polynomial of degree $2n$ of $\cos x$ at $x = 0$ is

$$P_{2n}(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \dots + (-1)^n \frac{1}{(2n)!} x^{2n}.$$

We also have $P_{2n+1} = P_{2n}$.

Example 4.38. The Taylor polynomial of degree $2n + 1$ of $\sin x$ at $x = 0$ is

$$P_{2n+1}(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots + (-1)^n \frac{1}{(2n + 1)!} x^{2n+1}.$$

We also have $P_{2n+2} = P_{2n+1}$.

Example 4.39. The Taylor polynomial of degree n of $1/x$ at $x = 1$ is

$$P_n(x) = 1 - (x-1) + (x-1)^2 - \cdots + (-1)^n (x-1)^n.$$

Example 4.40. The Taylor polynomial of degree n of $\log x$ at $x = 1$ is

$$P_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + (-1)^{n+1}(x-1)^n.$$

We write

$$f(x) = P_n(x) + R_n(x).$$

where R_n is the error, or remainder, between f and its Taylor polynomial P_n. The next theorem is one version of Taylor's theorem, which gives an expression for the remainder due to Lagrange. It can be regarded as a generalization of the mean value theorem, which corresponds to the case $n = 0$.

The proof is a bit tricky, but the essential idea is to subtract a suitable polynomial from the function and apply Rolle's theorem, just as we proved the mean value theorem by subtracting a suitable linear function.

Theorem 4.41 (Taylor)**.** *Suppose $f : (a,b) \to \mathbb{R}$ has $n+1$ derivatives on (a,b) and let $a < c < b$. For every $a < x < b$, there exists ξ between c and x such that*

$$f(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-c)^{n+1}.$$

Proof. Fix $x, c \in (a,b)$. For $t \in (a,b)$, let

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{1}{2!}f''(t)(x-t)^2 - \cdots - \frac{1}{n!}f^{(n)}(t)(x-t)^n.$$

Then $g(x) = 0$ and

$$g'(t) = -\frac{1}{n!}f^{(n+1)}(t)(x-t)^n.$$

Define

$$h(t) = g(t) - \left(\frac{x-t}{x-c}\right)^{n+1} g(c).$$

Then $h(c) = h(x) = 0$, so by Rolle's theorem, there exists a point ξ between c and x such that $h'(\xi) = 0$, which implies that

$$g'(\xi) + (n+1)\frac{(x-\xi)^n}{(x-c)^{n+1}} g(c) = 0.$$

It follows from the expression for g' that

$$\frac{1}{n!}f^{(n+1)}(\xi)(x-\xi)^n = (n+1)\frac{(x-\xi)^n}{(x-c)^{n+1}} g(c),$$

and using the expression for g in this equation, we get the result. □

Note that the remainder term

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1}$$

has the same form as the $(n+1)$th term in the Taylor polynomial of f, except that the derivative is evaluated at an (unknown) intermediate point ξ between c and x, instead of at c.

Example 4.42. Let us prove that

$$\lim_{x \to 0} \left(\frac{1 - \cos x}{x^2} \right) = \frac{1}{2}.$$

By Taylor's theorem,

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}(\cos \xi)x^4$$

for some ξ between 0 and x. It follows that for $x \neq 0$,

$$\frac{1 - \cos x}{x^2} - \frac{1}{2} = -\frac{1}{4!}(\cos \xi)x^2.$$

Since $|\cos \xi| \le 1$, we get

$$\left| \frac{1 - \cos x}{x^2} - \frac{1}{2} \right| \le \frac{1}{4!}x^2,$$

which implies that

$$\lim_{x \to 0} \left| \frac{1 - \cos x}{x^2} - \frac{1}{2} \right| = 0.$$

Note that Taylor's theorem not only proves the limit, but it also gives an explicit upper bound for the difference between $(1 - \cos x)/x^2$ and its limit $1/2$.