① $|x_{n+1} - x_n| \le 1/2^n$ for all $n \geq 1$.

Pick any ε. Note that for $m > n \geq N$,

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - \dots + x_{n+1} - x_n|$$
$$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$
$$\le 1/2^{m-1} + 1/2^{m-2} + \dots + 1/2^n$$
$$\le 1/2^n (1 + 1/2 + 1/2^2 + \dots) < \frac{1}{2^{n-1}} < \frac{1}{2^{N-1}}.$$

Pick N large enough so that $1/2^{N-1} < \varepsilon$.
We can find such N because $\lim_{n \to \infty} 1/2^{n-1} = 0$.

② Fix $\varepsilon > 0$.
$$|x_m - x_n| = \left| \sum_{k=n+1}^{m} \frac{\sin k}{k^2} \right| \le \sum_{k=n+1}^{m} \frac{|\sin k|}{k^2}$$

$$y_n = \sum_{k=1}^{n} 1/k^2 \qquad \le \sum_{k=n+1}^{m} \frac{1}{k^2} = |y_m - y_n|.$$

Converges $\Rightarrow$ Cauchy $\Rightarrow |y_m - y_n| < \varepsilon \ \forall m, n \geq$ some N.
$\Rightarrow |x_m - x_n| < \varepsilon \ \forall m, n \geq N$
$\Rightarrow x_n$ is cauchy seq.

③ No, counter-example: $x_n = \sqrt{n}$

④ No, counter-example: $x_n = 1/n$.

$x_n = \frac{1}{n}$. Let, if possible, $|x_{n+1} - x_n| \le \lambda\, |x_n - x_{n-1}|$

$$\Rightarrow \left|\frac{1}{n+1} - \frac{1}{n}\right| \le \lambda \left|\frac{1}{n} - \frac{1}{n-1}\right|$$

$$\Rightarrow \frac{1}{(n+1)\,n} \le \lambda \frac{1}{n\,(n-1)}$$

$$\Rightarrow \lambda \ge \frac{n-1}{n+1} \text{ for every } n \ge 1$$

(let $n \to \infty$) $\Rightarrow \lambda \ge 1 \longrightarrow$ Not allowed!

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- $1 \le x_n \le 2$ for every $n \ge 1$.
  ↳ easy by induction.

- $x_{n+1}/x_n \le x_{n+1} \le 2$, for every $n \ge 1$. (a)
  ($\because x_n \ge 1$) ($\because x_{n+1} \le 2$)

- $|x_{n+2} - x_{n+1}| = |\sqrt{x_{n+1}\, x_n} - x_{n+1}|$

$$= \sqrt{x_{n+1}}\, |\sqrt{x_n} - \sqrt{x_{n+1}}|$$

$$= \frac{\sqrt{x_{n+1}}}{\sqrt{x_n} + \sqrt{x_{n+1}}}\, |x_{n+1} - x_n|$$

$$= \frac{1}{1 + \sqrt{\frac{x_n}{x_{n+1}}}}\, |x_{n+1} - x_n|$$

$$\frac{x_{n+1}}{x_n} \le 2 \Rightarrow \sqrt{\frac{x_n}{x_{n+1}}} \ge \frac{1}{\sqrt{2}} \ge \frac{1}{2}$$

$$\le \frac{1}{1 + 1/2}\, |x_{n+1} - x_n|$$

$$= 2/3\, |x_{n+1} - x_n|.$$

Hence $\lim_{n \to \infty} x_n$ exists.

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