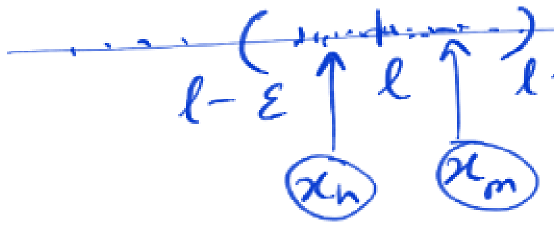
Cauchy Sequence

$x_n \to \ell$ as $n \to \infty$



close to each other

If $|x_n - x| < \varepsilon$ holds for every $n \geq N$, then for any $m, n \geq N$, we have $|x_n - x_m| \leq 2\varepsilon$.

Q Suppose for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. for every $n \geq N$, $|x_{n+1} - x_n| < \varepsilon$ holds.

$\to$ Does this imply that x_n converges?

A: NO. For instance, $x_n = \sqrt{n}$ has this property.

$$x_n = \sqrt{n} \to x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon \text{ for all } n \geq \text{some } N.$$

But $\lim x_n$ does not exist.

x_{n+1}, x_n getting closer and closer $\to$ Not enough

x_n, x_m getting closer and closer

$\to$ will this be enough? Let's see.

Def. (Cauchy sequence)

A sequence $\{x_n\}_{n \geq 1}$ is called a Cauchy sequence if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. for every $n, m \geq N$,

$$|x_n - x_m| < \varepsilon \text{ holds.}$$

One Food for thought

$\lim_{n\to\infty}(x_{n+1}-x_n)=0$. But $\lim_{n\to\infty} x_n$ does not exist.

$? \Rightarrow \lim_{n\to\infty} x_n = +\infty$ or $-\infty$.

Definition of Cauchy Sequence $\rightarrow$ we learnt.

Examples? $\frac{1}{n}$, $\frac{1}{\sqrt{n}}$, $2+\frac{1}{n}$ $\rightarrow$ these are Cauchy (check)

In fact, any convergent sequence is also a Cauchy sequence. Because

$|x_n-\ell|<\varepsilon/2$ for all $n\geq N$ $\Rightarrow$ $|x_n-x_m|\leq|x_n-\ell|+|x_m-\ell|$
for any $m,n\geq N$ $\leq \varepsilon/2+\quad=\varepsilon$

Is the converse also true? That is,

Cauchy $\overset{?}{\Rightarrow}$ Convergent

$\boxed{1}$ Cauchy $\Rightarrow$ Bounded

$|x_n|-|x_m|\leq|x_n-x_m|\leq 1$, for all $m,n\geq N$

$\Rightarrow |x_n|\leq |x_m|+1$, for all $m,n\geq N$.

Let's fix $m=N$. Then we get $|x_n|\leq|x_N|+1$ for every $n\geq N$.

Now take $M=\max\{|x_1|,\dots,|x_{N-1}|,|x_N|+1\}$.

Then $|x_n|\leq M$ for every $n\geq 1$.

$\boxed{2}$ Cauchy $\Rightarrow$ Bounded $\Rightarrow$ Has a convergent subsequence (by BW thm)

$\Downarrow$

Convergent

Intuition / Proof

Let x_n be Cauchy. $\Rightarrow$ x_n bdd

(BW) $\Rightarrow$ It has a conv. subseq. say x_{n_k}.

Suppose $x_{n_k} \to \ell$, as $k \to \infty$.

" $x_{n_k} \to \ell$ "

$$|x_{n_k} - \ell| < \varepsilon \quad (\dagger)$$

for all suff. large k

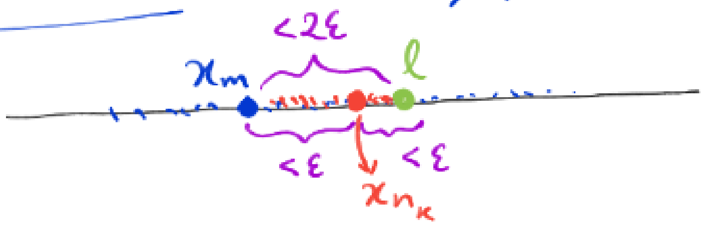
say for all $k \geq K_\varepsilon$

" x_n is Cauchy "

$$|x_n - x_m| < \varepsilon$$

for all sufficiently large n & m. $(*)$

Say for all $n, m \geq N$.



Fix $\varepsilon > 0$.

First choose N s.t. $(*)$ holds, and K s.t. $(\dagger)$ holds.

Now, for some $k \geq K$, n_k will be greater than N also.

Fix $n = n_k$ in $(*)$ and conclude that for every $m \geq N$, $|x_{n_k} - x_m| < \varepsilon$.

Since $k \geq K$, we also have $|x_{n_k} - \ell| < \varepsilon$.

Therefore, for every $m \geq N$

$$|x_m - \ell| \leq |x_m - x_{n_k}| + |x_{n_k} - \ell| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we are through.

Convergent $\iff$ Cauchy $\to$ Then why care about "Cauchy sequences"?

It is yet another tool for proving convergence without guessing the limit. It works also for sequences that are not monotone and when it is not easy to give bounds in order to apply Sandwich!

Example Suppose $|x_{n+1} - x_n| \le \lambda |x_n - x_{n-1}|$ holds for all $n > 1$, where $\lambda \in (0,1)$. Show that $\{x_n\}$ is Cauchy.

$$|x_{n+1} - x_n| \le \lambda^1 |x_n - x_{n-1}| \le \lambda^2 |x_{n-1} - x_{n-2}|$$
$$\cdots \le \lambda^{n-1} |x_2 - x_1|, \text{ for all } n \ge 1.$$

Now, for any $n > m$,

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \ldots + x_{m+1} - x_m|$$
$$\le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{m+1} - x_m|$$
$$\le (\lambda^{n-2} + \lambda^{n-3} + \ldots + \lambda^{m-1}) |x_2 - x_1|$$
$$\le \lambda^{m-1} |x_2 - x_1| (1 + \lambda + \lambda^2 + \ldots)$$
$$= \lambda^{m-1} |x_2 - x_1| \frac{1}{1-\lambda} = \boxed{\lambda^m} \underbrace{\frac{|x_2 - x_1|}{\lambda(1-\lambda)}}_{\text{Const say } c}$$
$\le \varepsilon/c$ eventually.

So, $\boxed{\text{take } N \text{ s.t. } \lambda^N < \varepsilon/c}$

Then for any $n > m \ge N$, we would have
$$|x_n - x_m| \le \lambda^m c \le \lambda^N c < \varepsilon.$$

Example $a_1 = 1$, $a_{n+1} = 1 + 1/a_n$, $n \ge 1$.

$\lim_{n \to \infty} a_n = ?$

Hint $|a_{n+1} - a_n| < \lambda |a_n - a_{n-1}|$ ✓ Prove!

Find $\lambda \in (0,1)$ s.t. it occurs.

Also, try the exercises given in the note.