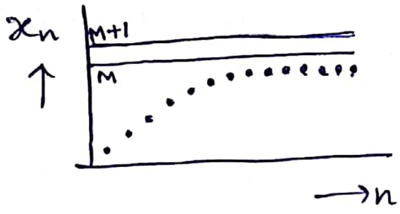
2. Convergence of Monotone Sequences

Suppose $\{x_n\}_{n\geq 1}$ is an increasing sequence. If $\{x_n\}_{n\geq 1}$ is not bounded above (for example, $x_n = n$) then it does not converge. But if $\{x_n\}_{n\geq 1}$ is bounded above, that is $x_n \leq M$ holds for all $n \geq 1$, where M is fixed, then can we conclude anything?



The information that we have in our hand, is

(i) $x_{n+1} \geq x_n$ for each $n \geq 1$.

and (ii) $x_n \leq M$ for each $n \geq 1$.

It seems that $\{x_n\}_{n\geq 1}$ converges to some number, which need not be M. Because if $x_n \leq M$ holds for each $n \geq 1$, then $x_n \leq M+1$ holds as well. And x_n can't converge to both M and M+1.

Looking at the figure above, it seems that the sequence $\{x_n\}_{n\geq 1}$ converges to its 'least' upper bound, if there is any such number. We define it as follows:

Definition (least upper bound/supremum)

Suppose A is a non-empty subset of $\mathbb{R}$, such that the set is bounded ~~from~~ above. We say that l is a least upper bound for A if

(i) l is itself an upper bound, that is, $a \leq l$ holds for all $a \in A$.

(ii) If $l' < l$ then l' is not an upper bound. Means, for every $l' < l$, there exists $a^* \in A$ such that $a^* > l'$.

Note, we can rewrite (ii) as —

(ii)' For every $\varepsilon > 0$, there exists $a^* \in A$ such that $a^* > l - \varepsilon$.

Also note that if a least upper bound exists, then it must be unique. That is, a set can't have two different least upper bounds (Prove it yourself).

But, how to ensure that a set has least upper bound? If the set is not bounded ~~fro~~ above, then clearly it can't have a least upper bound. But, if a set is bounded above, is it necessary that it has a 'least' upper bound?

The answer turns out to be 'Yes', but ~~not~~ in an unexpected manner. It is actually an axiom for $\mathbb{R}$, and in fact this is the only axiom that separates $\mathbb{R}$ from $\mathbb{Q}$.

> <u>Axiom</u> (Axiom of Completeness)
>
> If a non-empty subset A of $\mathbb{R}$ is bounded above, then it has a least upper bound.

This axiom has several nice applications, but it is beyond our scope to dig into that treasure. Interested reader can look up any book on Real Analysis (for example, Rudin or Apostol).

One important thing to note is that, we don't need another axiom for sets bounded from below to have a greatest lower bound (~~infinimum~~ infimum). Because, if A is bounded from below, then $-A=\{-a : a \in A\}$ is bounded from above, so it has a supremum, say l. Then, $-l$ will be a greatest lower bound (infimum) for A. One can also show that infimum of a set must be unique, if it exists.

Theorem If $\{x_n\}_{n\geq 1}$ is bounded from above and increasing, then it converges.

Proof. The set $\{x_n : n \geq 1\}$ is bounded above, so it has a least upper bound, say l. Then from definition, (i) $x_n \leq l$ holds for all $n \geq 1$, and (ii) for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_N > l - \varepsilon$.

Fix any $\varepsilon > 0$. Since there exists $N \in \mathbb{N}$ such that $x_N > l - \varepsilon$ and $\{x_n\}_{n\geq 1}$ is increasing, so for every $n \geq N$, we have $l - \varepsilon < x_N \leq x_n < l$.

Therefore, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $x_n \in (l-\varepsilon, l+\varepsilon)$.

Hence, x_n converges to l, as $n \to \infty$. $\square$

Note, if $\{x_n\}_{n\geq 1}$ is increasing, then it is automatically bounded below, because $x_n \geq x_1$ for all $n \geq 1$.

So, we can restate the above theorem as — "Every increasing and bounded sequence must converge".

Similarly, we have a theorem for decreasing and bounded sequences —

Theorem If $\{x_n\}_{n\geq 1}$ is decreasing and bounded below, then ~~show that~~ x_n converges.

Proof: Do it yourself. You can mimic the above proof for increasing sequences. Here you have to use greatest lower bound/infimum. Alternately, you can apply the previous theorem to $\{-x_n\}_{n\geq 1}$.

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Example. Let $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$, $n \geq 1$. Is it true that x_n converges?

Solution $x_{n+1} - x_n = \frac{1}{(n+1)^2} > 0$, for each $n \geq 1$.

Therefore the sequence is increasing.

Next, $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(n-1)\cdot n}$

$= 2 - \frac{1}{n}$ (Show)

< 2, for each $n \geq 1$.

Thus, $\{x_n\}_{n\geq 1}$ is increasing and bounded, hence it must converge. [Since $x_n < 2$ for each $n \geq 1$, we can also say that $\lim_{n\to\infty} x_n \leq 2$. However, the theorem that we used, does not give us the value of $\lim_{n\to\infty} x_n$.]

Example. Suppose $\{a_n\}_{n\geq 1}$ is a sequence satisfying

$$a_{n+1} = \frac{3a_n}{2+a_n}, \text{ for each } n \geq 1.$$

(a) If $0 < a_1 < 1$, then show that the sequence a_n is increasing and hence show that $\lim_{n\to\infty} a_n = 1$.

(b) If $a_1 > 1$, then show that the sequence a_n is decreasing, and hence show that $\lim_{n\to\infty} a_n = 1$.

(c) What happens if $a_1 = 1$?

Solution (a) First we show that $0 < a_n < 1$ holds for all $n \geq 1$. We show it by induction on n. The base case $n=1$ is given to be true. And the inductive step (from n to $n+1$) is as follows: ~~($a_n > 0$ can be seen easily. We need to show the other side only.)~~ $a_n > 0 \Rightarrow a_{n+1} > 0$. And,

$$a_{n+1} = \frac{3a_n}{2+a_n} \leq \frac{2+a_n}{2+a_n} = 1.$$ [We used $0 < a_n < 1$ to get the inequality.]

Next, we show that a_n is increasing.

Note that $a_{n+1} - a_n = \dfrac{3a_n}{2+a_n} - a_n = \dfrac{a_n(1-a_n)}{2+a_n} > 0$, because

$0 < a_n < 1$. ~~Since this holds for~~

Therefore, $a_{n+1} > a_n$ for all $n \geq 1$, and $0 < a_n < 1$ for all $n \geq 1$. Hence $\lim\limits_{n\to\infty} a_n$ exists. Say $l = \lim\limits_{n\to\infty} a_n$.

Since $a_{n+1} = \dfrac{3a_n}{2+a_n}$ for all $n \geq 1$, we let $n \to \infty$ to obtain $l = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{3a_n}{2+a_n} = \dfrac{3l}{2+l}$.

[Note, $0 < a_n < 1 \Rightarrow 0 \leq l \leq 1$, so $2+l \neq 0$, $2+a_n \neq 0$ for each $n \geq 1$.]

Solving for l, we get $l = 0$ or 1.

Now, the sequence is increasing, so $a_n \geq a_1 > 0$ for all $n \geq 1$, which gives $l \geq a_1 > 0$. Hence $l \neq 0$, i.e. $\boxed{l = 1}$.

(b) Do it yourself. Its very similar to (a).
[First show that $a_n > 1$ holds for all $n \geq 1$. Then show that a_n is decreasing. Thus a_n is decreasing and bounded below, so $\lim\limits_{n\to\infty} a_n = l$ exists. Solve for l and eliminate the possibility that $l = 0$.]

(c) If $a_1 = 1$, then $a_n = 1$ for each $n \geq 1$. □

Exercises

2.7. Suppose $a_{n+1} = a_n^2$ for all $n \geq 1$ and $0 < a_1 < 1$. Show that $\lim\limits_{n\to\infty} a_n$ exists. Find this limit.

2.8. Suppose $\{x_n\}_{n\geq 1}$ satisfies $x_{n+1} = x_n(2 - x_n)$, for every $n \geq 1$. If $0 < x_1 < 1$, then show that the sequence x_n converges. Also, find its limit.

2.9. Define $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for each $n \geq 1$. Does $\lim\limits_{n\to\infty} x_n$ exist? If yes, then find it.

2.10. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$, for $n \geq 1$, where $\alpha > 0$. If $x_1 \geq \sqrt{\alpha}$, then show that x_n converges. Also, find ~~the~~ $\lim_{n\to\infty} x_n$.

2.11. Suppose a calculator can do only addition, subtraction, multiplication and division. (In particular, it does not have buttons for square-root or to-the-power.) Find a way to approximate $\sqrt{5}$ upto few places of decimal.

2.12. Suppose that $\{S_n\}_{n=1}^{\infty}$ satisfies $S_{n+1} = \sqrt{\frac{ab^2 + S_n^2}{a+1}}$, for each $n \geq 1$, where $b > a > 0$. And let $S_1 = a$. Find whether $\{S_n\}_{n=1}^{\infty}$ converges or not. (If it converges, then find its limit too.)

2.13. Suppose $\{x_n\}_{n\geq 1}$ is a bounded sequence. Show that it has a subsequence which converges. [Hint: Combine two theorems we learnt today.]

2.14. Suppose that $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are both bounded. Show that there exists a sequence $1 \leq n_1 < n_2 < n_3 < \cdots$ of positive integers, such that both $\{x_{n_k}\}_{k\geq 1}$ and $\{y_{n_k}\}_{k\geq 1}$ converges.

2.15. Does there exist an unbounded sequence which does not have any convergent subsequence?

2.16. Does there exist an unbounded sequence which has a convergent subsequence?

2.17. Suppose we have a sequence $\{x_n\}_{n\geq 1}$ such that every convergent subsequence of it converges to same limit (say l). If $\{x_n\}_{n\geq 1}$ is bounded, is it necessary that x_n converges to l? (Compare this with exercise 2.4.)

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