Exercises on monotone sequences (Solutions)

2.7 $a_{n+1} = a_n^2$, $0 < a_1 < 1$.

$0 < a_{n+1} < a_n \rightarrow$ prove by induction

$\therefore$ $a_n$ is decreasing and bounded below by 0,

so $\lim_{n\to\infty} a_n$ must exist. Let $\lim_{n\to\infty} a_n = l$. Now,

$a_{n+1} = a_n^2$ for all $n \geq 1$. letting $n \to \infty$ here,

we get $l = l^2 \Rightarrow l = 0, 1$.

Since $a_{n+1} \leq a_1 \Rightarrow l \leq a_1 < 1$. So $l = 0$.

2.8. $x_{n+1} = x_n(2 - x_n)$, $0 < x_1 < 1$.

$\frac{x_2}{x_1} = 2 - x_1 > 1$ ($\because 0 < x_1 < 1$)

claim $0 < x_n < 1$ and $x_{n+1} > x_n$ for all $n \geq 1$.

$$x_{n+1} = 2x_n - x_n^2$$

$$\Rightarrow 1 - x_{n+1} = (1 - x_n)^2$$

$a_n = 1 - x_n \rightarrow$ then it is same as the prev. problem.

In prev problem, we saw that $a_n$ was dec.

So here $x_n$ will be inc. and bounded above by 1.

So $x_n$ converges.

Now find the limit from the recursion.

$$x_{n+1} = x_n(2 - x_n) \xrightarrow{n\to\infty} l = l(2 - l)$$

($l \neq 0$ since positive & inc) $\Rightarrow 1 = 2 - l \Rightarrow l = 1$.

2.9. $x_{n+1} = \sqrt{2x_n}$, $n \geq 1$, $x_1 = \sqrt{2}$.

- $0 < x_n < 2$ and $x_n < x_{n+1}$ for all $n \geq 1$
- Hence $\lim_{n\to\infty} x_n$ exists. Call it $l$.
- $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n} \Rightarrow l = \sqrt{2l} \Rightarrow l = 2.$

  ($l \neq 0 \because$ inc and positive)

2.10. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$, $\alpha > 0$, $n \geq 1$. $x_1 \geq \sqrt{\alpha}$.

By AM-GM, $x_n \geq \sqrt{\alpha}$ for all $n \geq 1$.

$$x_{n+1}/x_n = \frac{1}{2}\left(1 + \alpha/x_n^2\right) \leq 1$$

$\Rightarrow x_n$ is dec. and bounded below by $\sqrt{\alpha}$.

$\Rightarrow x_n$ converges. Let $\lim_{n\to\infty} x_n = l$.

Now, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \overset{n\to\infty}{\Longrightarrow} l = \frac{1}{2}\left(l + \frac{\alpha}{l}\right)$

$\Rightarrow l = \pm\sqrt{\alpha}$.

Since $x_n \geq \sqrt{\alpha}$ for all $n$, so $l = \sqrt{\alpha}$.

2.11. Define a sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right), \quad n \geq 1, \quad x_1 = 1.$$

a = 1

Repeat [ ANS + 5/ANS
ANS/2 ]

Try with a hand-held calculator.

ANS $\to$ converges to $\sqrt{5}$

2.12. $S_{n+1} = \sqrt{\frac{ab^2 + S_n^2}{a+1}}$, $n \geq 1$, $b > a > 0$. $S_1 = a$.

$0 < S_n < b$.
(by induction) $S_{n+1} = \sqrt{\frac{ab^2 + S_n^2}{a+1}} < \sqrt{\frac{ab^2 + b^2}{a+1}} = b.$

$$\left(S_{n+1}/S_n\right)^2 = \frac{a}{a+1}\left(\frac{b}{S_n}\right)^2 + \frac{1}{a+1} \geq \frac{a}{a+1} + \frac{1}{a+1} = 1$$

$\Rightarrow S_{n+1} \geq S_n$. So $S_n$ is inc and bdd above by $b$.

$\Rightarrow$ Converges.

Let $l = \lim_{n\to\infty} S_n$.

$$S_{n+1}^2 = \frac{ab^2 + S_n^2}{a+1} \underset{(n\to\infty)}{\Rightarrow} l^2 = \frac{ab^2 + l^2}{a+1}$$

$$\Rightarrow l = \pm b \Rightarrow \underline{l = b \ (\because l \geq 0)}$$

2.13. (Bolzano-Weierstrass theorem)

Suppose $x_n$ is bounded. Now $x_n$ has a monotone subseq., say $x_{n_k}$, which is also bounded. Hence the subseq. $x_{n_k}$ converges.

2.14 $\{x_n\}$, $\{y_n\}$ both bounded.

Show that $\exists\ n_1 < n_2 < n_3 < \cdots$ s.t. $\{x_{n_k}\}$ and $\{y_{n_k}\}$ both conv.

$x_1 \quad x_2 \quad \cdots$

$y_1 \quad y_2 \quad \cdots$

no connection

$\{x_n\}$ bdd $\Rightarrow \exists$ a subseq $\{x_{m_k}\}_{k\geq 1}$ which converges.

$\{y_{m_k}\}_{k\geq 1}$ bdd $\Rightarrow \exists$ a subseq $\{y_{m_{k_l}}\}_{l\geq 1}$ which converges.

Call $n_l = m_{k_l}$, $l \geq 1$.

Then $\{y_{n_l}\}$ converges and $\{x_{n_l}\}$ also converges, being a subseq. of a convergent sequence.

$x_1, x_2, \ldots \overset{\exists}{\Rightarrow} x_{m_1}, x_{m_2}, \ldots \Rightarrow x_{m_{k_1}}, x_{m_{k_2}}, \ldots$

bdd — converges — converges

$y_{m_1}, y_{m_2}, \ldots \overset{\exists}{\Rightarrow} y_{m_{k_1}}, y_{m_{k_2}}, \ldots \leftarrow$ (same indices)

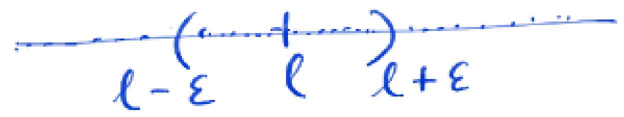
bdd — converges

2.15 $1, 2, 3, 4, 5, \ldots$

2.16. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$

↳ conv. subseq: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$

2.17. $\{x_n\}$ is a seq. such that every conv. subseq. of $\{x_n\}$ converges to $l$. Exc 2.4. tells us that this is not sufficient to conclude that $\lim_{n\to\infty} x_n$ exists. However, in this problem, we have another assumption that $\{x_n\}$ bdd. Can we tell now that $\lim x_n$ exists?




Since all conv. subsequences converge to $l$, the whole must also conv. to $l$ if it converges at all.

Let, if possible, $x_n$ do $\boxed{\text{not}}$ converge to $l$.

$\varepsilon$ def.

$\exists\, \varepsilon > 0$ s.t. for any $N \in \mathbb{N}$, $\exists\, n \geq N$ s.t. $|x_n - l| > \varepsilon$.

$\Downarrow$

$\exists\, \varepsilon > 0$ s.t. $\exists$ a subseq. $x_{n_k}$ that lies outside $(l-\varepsilon, l+\varepsilon)$.

Now this subseq $\{x_{n_k} : k \geq 1\}$ is bdd.

$\Downarrow$ Bolzano-Weierstrass

It has a convergent subseq., say $\{x_{n_{k_l}} : l \geq 1\}$.

But that is not possible, because the seq. $x_{n_k}$ completely lies outside $(l-\varepsilon, l+\varepsilon)$. Contradiction.