More Problems on Sequences and Series

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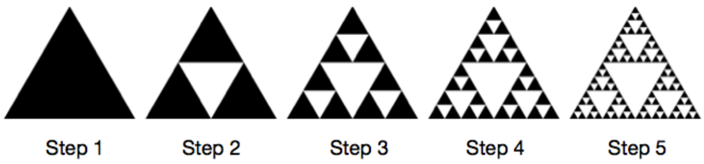
1. Suppose x_n is a sequence that converges to x. Show that $\sin x_n$ converges to $\sin x$ and $\cos x_n$ converges to $\cos x$. Does it imply that $\tan x_n$ converges to $\tan x$? (You may use the result that $|\sin t| \leq |t|$ holds for every $t \in \mathbb{R}$.)

2. Define $x_1 = \sqrt{2}$, $x_2 = \sqrt{2}^{\sqrt{2}}$, $x_3 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$ and so on. We want to find $\lim\limits_{n\to\infty} x_n$. We set the limit to be x and arrive at $x = \sqrt{2}^{\,x}$. Now, $x = 2$ and $x = 4$ both satisfy the last equation. Which one of these is the correct value of the limit?

3. Use trigonometry to find the perimeter and the area of a regular n-gon inscribed in a circle of unit radius. If this perimeter be P_n and this area be A_n, find the limit of P_n and A_n as $n \to \infty$. Does the result seem intuitive?

4. Start with an equilateral triangle with unit side length. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle. Repeat last step with each of the remaining smaller triangles.



   Denote by $P(n)$ and $A(n)$ the perimeter and area of the existing portion of the triangle at the n-th step, e.g. $P(2) = 9/2$ unit and $A(2) = 3\sqrt{3}/16$ sq.unit. Find $\lim\limits_{n\to\infty} P(n)$ and $\lim\limits_{n\to\infty} A(n)$. Are you surprised?

5. Suppose $\{x_n\}_{n\geq 1}$ is a sequence such that $0 < x_n < 1$ holds for each $n \geq 1$. Furthermore, suppose we have $4x_n(1 - x_{n+1}) > 1$ for every $n \geq 1$. Show that x_n converges and also find the limit.

6. Suppose $\{x_n\}_{n\geq 1}$ is a sequence such that

$$x_2 \leq x_4 \leq x_6 \leq \cdots \leq x_5 \leq x_3 \leq x_1.$$

   Define $y_n = x_{2n-1} - x_{2n}$ for all $n \geq 1$. Assume further that $\lim\limits_{n\to\infty} y_n = 0$. Show that $\{x_n\}_{n\geq 1}$ must converge. (You are neither asked, nor able, to find the limit of x_n.)

7. Calculate the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\cos^3(3^n x)}{3^n}$. (Hint: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

8. (a) Find the largest positive integer not exceeding the value of $\displaystyle\sum_{k=1}^{1599} \frac{1}{\sqrt{k}}$.

   (b) Determine, with proof, the value of $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}}\Big(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\Big)$.

9. (a) Show that $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}^{-1}$ exists and also evaluate this limit.

   (b) Show that $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} k\binom{n}{k}^{-1} = +\infty$ (in the sense that it diverges to $+\infty$).

10. For $m \in \mathbb{N}$, define $f_m(n) = 1^m + 2^m + \cdots + n^m$. Show that f_m is a polynomial in n and find its degree and leading coefficient. Hence evaluate the limit

$$\lim_{n\to\infty} \frac{1^m + 2^m + \cdots + n^m}{n^{m+1}}.$$

   Hint: Use $(k+1)^{m+1} - k^{m+1} = (m+1)k^m + \binom{m+1}{2}k^{m-1} + \cdots + \binom{m+1}{m+1}$. Sum it up for $k = 1, 2, \ldots, n$. And use induction on m.

11. Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Define another sequence $\{a_n\}_{n\geq 1}$ as

$$a_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n),\ n \geq 1.$$

   (a) If x_n is bounded, show that a_n must be bounded too. Is the converse true?

   (b) If x_n converges to ℓ, show that a_n must also converge to ℓ. Is the converse true?

   (c) Suppose that a_n converges to ℓ. Does this imply that x_n is at least bounded?

   Comment: This is really an important result. We often use part (b) to calculate limits.

12. Define $a_n = (2^n + 3^n + 6^n)^{1/n}$, $n \geq 1$. Calculate $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k$, and $\displaystyle\lim_{n\to\infty} \frac{n}{\sum_{k=1}^{n} 1/a_k}$.

13. Suppose that $\{a_n\}_{n\geq 0}$ is a sequence of positive reals which is bounded. Show that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0 \iff \lim_{n\to\infty} \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} = 0.$$

14. If $\{a_n\}_{n\geq 1}$ be a positive sequence such that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, is it necessary that $\displaystyle\lim_{n\to\infty} na_n = 0$?

   Hint: We have seen this to hold if a_n is a decreasing sequence. Try to tweak a convergent series, such as $\sum 1/n^2$, to see if you can get any counter-example.

15. Let $\{a_n\}_{n\geq 1}$ be a positive sequence such that $\sum_{n=1}^{\infty} a_n$ diverges. Show that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

    Hint: Consider two cases: whether a_n is bounded or unbounded. Treat them separately.

16. Let a_n be a decreasing sequence of positive real numbers. Show that

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}$$

    Hint: Since both are series of positive numbers, it is enough to show that they are bounded. Using the decreasing nature of a_n's to compare $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} 2^n a_{2^n}$.

17. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges. What about $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$?

18. Let x_n be a sequence of integers such that $x_{k+1} \neq x_k$ holds for every $k \geq 1$. Show that x_n can not be a Cauchy sequence. Is it possible that x_n has a convergent subsequence?

19. A Physics problem demands you to evaluate the following continued fraction

$$a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{\ddots}}}.$$

    The usual way of doing this is to assume it equals x and then solving the equation $x = a + \frac{b}{x}$. But, after learning about some fallacies in series, we are now skeptical of "assuming it equals x". So we aim to show formally that the above continued fraction "converges", only then assume it equals x.

    (a) Fix any $a, b > 0$. Define $x_{n+1} = a + \frac{b}{x_n}$ for every $n \geq 0$. Take any value of $x_0 > 0$. Show that $\lim_{n\to\infty} x_n$ exists. Also, find the limit.
    Hint: First, $x_n \geq a$ for all $n \geq 1$. Now show that $|x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}|$ holds for every $n \geq 1$, where $\lambda \in (0, 1)$ is a constant. (Find a suitable λ here!)

    (b) Check whether the limit does not exist for some $a < 0$ or $b < 0$. (In Physics you usually have $a, b > 0$.)