Problems on Series

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1. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)(n+4)}$ converges and also find its sum.

2. Evaluate the series: $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{(n-1)!+n!+(n+1)!}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}$.

3. Evaluate the series:

   (a) $1+\frac{1}{2}+\frac{1}{4^2}+\frac{1}{2^3}+\frac{1}{4^4}+\frac{1}{2^5}+\frac{1}{4^6}+\cdots$.

   (b) $1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2^2\cdot 3}+\frac{1}{2^2\cdot 3^2}+\frac{1}{2^3\cdot 3^2}+\frac{1}{2^3\cdot 3^3}+\cdots$.

4. Suppose that $a_{n+2}=a_{n+1}+a_n$ for all $n\geq 1$ and $a_1=1, a_2=2$. Evaluate the series
$$\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+2}}.$$
   Comment: Note that a_n is essentially the sequence of Fibonacci numbers. However, you don't need any result on Fibonacci numbers to do this problem.

5. Show that $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{1+n+n^2}\right)$ converges and also find its sum.

6. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \sin(\pi\sqrt{n^2+n+1})$ converges or not.

7. Let $1,2,3,4,5,6,7,8,9,11,12,\cdots$ be the sequence of all positive integers which do not contain the digit zero. If we denote this sequence by $\{a_n\}_{n\geq 1}$, show that $\sum_{n=1}^{\infty} 1/a_n \leq 90$.

8. Show that the series
$$\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\cdots+\frac{2^n}{1+x^{2^n}}+\cdots$$
   converges when $|x|>1$, and in this case evaluate it.

9. If $\sum_{n=1}^{\infty} |x_{n+1} - x_n|$ converges, show that $\lim_{n\to\infty} x_n$ exists.

10. Show that for every $a < 1$, $\sum_{n=1}^{\infty} \frac{1}{n^a}$ diverges. (You may use that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.)

11. Suppose that $\sum_{n\geq 1} |a_n|$ converges. Consider the series $\sum_{n\geq 1}(a_n + |a_n|)$. This is a series of non-negative terms. Give an upper bound on its partial sum to show that it converges. Hence conclude that $\sum_{n\geq 1} a_n$ converges.

12. (a) For any $x \in \mathbb{R}$, define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Show that for every $x \in \mathbb{R}$, $x = x^+ - x^-$ and $|x| = x^+ + x^-$. (Hint: Split into two cases as $x \geq 0$ or $x < 0$.)

    (b) Show that $\sum_{n\geq 1} |a_n|$ converges if and only if $\sum_{n\geq 1} a_n^+$ and $\sum_{n\geq 1} a_n^-$ are convergent.

13. Suppose that a_n is a sequence that decreases to zero (i.e. a_n is decreasing and converges to 0). Show that the 'alternating' series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges. Hence conclude that the series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges.

    Hint: Consider the sequence of partial sums $S_n = \sum_{k=1}^{n}(-1)^{k-1}a_k$. Show that the sequence $\{S_{2n}\}_{n\geq 1}$ is increasing and bounded above by S_1 and $\{S_{2n+1}\}_{n\geq 1}$ is decreasing and bounded above by S_2. Finally, use $S_{2n+1} - S_{2n} = a_{2n+1}$ to show that these two subsequences converge to the same limit.

14. Suppose that $\{a_n\}_{n\geq 1}$ is a decreasing sequence of non-negative numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Show that $\lim_{n\to\infty} na_n = 0$.

    Hint: Let $S_n = \sum_{k=1}^{n} a_k$. Give an upper bound on a_{2n} using S_{2n} and S_n. Do similar thing for a_{2n+1}.

15. Let a_0, a_1 be any positive real number and define $a_{n+1} = \dfrac{na_n + a_{n-1}}{n+1}$ for all $n \geq 1$. Show that the sequence a_n converges.

    Hint: Give an upper bound on $|a_{n+1} - a_n|$ and show that $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ converges.

16. Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of non-negative numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Then show that $\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n}$ must also converge.

17. Let $a_1 = 3$ and $a_{n+1} = (a_n^2 + 1)/2$ for all $n \geq 1$. Show that for every $n \in \mathbb{N}$,

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_n + 1} + \frac{1}{a_{n+1} - 1} = \frac{1}{2}.$$

Hence show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n + 1} = \frac{1}{2}$.

18. For any positive integer n, let $\langle n \rangle$ denote the integer closest to $\sqrt{n}$. Evaluate the series

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

Hint: For a fixed $k \in \mathbb{N}$, find all $n \in \mathbb{N}$ which satisfy $\langle n \rangle = k$.

19. Let $x_0 = a, x_1 = b$ and define

$$x_{n+1} = \left(1 - \frac{1}{2n}\right) x_n + \frac{1}{2n} x_{n-1}$$

for $n \geq 1$. Show that $\lim_{n \to \infty} x_n$ exists and also find this limit.

20. (a) Start with de Moivre's theorem: $(\cos\theta + i \sin\theta)^m = (\cos m\theta + i \sin m\theta)$. Put $m = 2n + 1$ and $\theta = \frac{k\pi}{2n+1}$ and hence obtain a polynomial in $\cot^2\theta$ which vanishes when $\theta = \frac{k\pi}{2n+1}$, $k = 1, \ldots, n$.

(b) Using the polynomial in part (a), show that

$$\sum_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} = \frac{n(2n^2 - 1)}{3}.$$

(c) Use the inequality $\sin x < x < \tan x$ for $x \in (0, \pi/2)$, to derive lower and upper bounds on $\sum_{k=1}^{n} 1/k^2$ (also, use part (b)). Hence show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$