Sandwich

1. $\{\sqrt{n^2+n+1}\} = \sqrt{n^2+n+1} - \underbrace{[\sqrt{n^2+n+1}]}_{\text{equals } n}$

$n^2 < n^2+n+1 < (n+1)^2 \Rightarrow$

So, $\{\sqrt{n^2+n+1}\} = \sqrt{n^2+n+1} - n = \dfrac{n+1}{\sqrt{n^2+n+1}+n}$.

$$= \frac{1+1/n}{\sqrt{1+1/n+1/n^2}+1} \to \frac{1}{2}.$$

Alt

$n < \sqrt{n^2+n+1} < n+1 \Rightarrow \dfrac{n+1}{2n+1} < \dfrac{n+1}{\sqrt{n^2+n+1}+n} < \dfrac{n+1}{2n}$

By Sandwich, required limit is $1/2$.

2. $kx-1 < [kx] \le kx \Rightarrow \sum_{k=1}^{n}(kx-1) \le \sum_{k=1}^{n}[kx] \le \sum_{k=1}^{n} kx$

$$\Rightarrow x\,\frac{n(n+1)}{2} - n \le \sum_{k=1}^{n}[kx] \le x\,\frac{n(n+1)}{2}$$

Now divide by n^2 and apply Sandwich.

Limit $= \frac{x}{2}$

3. $f(n)$ = no. of digits of 8^n in base 6.

[If N has k digits in base b, then

$$N = (a_k a_{k-1} \cdots a_1)_b$$

$a_1, \ldots, a_k$ digits in base b $\in \{0, 1, \ldots, b-1\}$

$$= \underline{a_k \times b^{k-1}} + a_{k-1}\times b^{k-2} + \ldots + a_2 \times b + a_1$$

$$b^{k-1} \le N \le (b-1)\underline{(b^{k-1}+b^{k-2}+\ldots+b+1)}$$

$$= b^k - 1 < b^k$$

$\Rightarrow k-1 \le \log_b N < k \Longrightarrow \underline{k = [\log_b N]+1}$.]

$f(n) = \#$ digits of 8^n in base 6

$$= [\log_6 8^n] + 1 = [n \log_6 8] + 1.$$

$g(n) = \#$ digits of 6^n in base 8

$$= [\log_8 6^n] + 1 = [n \log_8 6] + 1.$$

$$\frac{f(n)\,g(n)}{n^2} = \frac{[n \log_6 8] + 1}{n} \times \frac{[n \log_8 6] + 1}{n}$$

$$\frac{(n \log_6 8)(n \log_8 6)}{n^2} < \downarrow < \frac{(n \log_6 8 + 1)(n \log_8 6 + 1)}{n^2}$$

By Sandwich, the required limit is $\log_6 8 \times \log_8 6 = 1$.

4. $0 \leq a_1 \leq a_2 \leq \cdots \leq a_r$

$$a_r^n \leq (a_1^n + a_2^n + \ldots + \underline{a_r^n}) \leq r \times a_r^n$$

$$\Rightarrow a_r \leq (a_1^n + a_2^n + \ldots + a_r^n)^{1/n} \leq \boxed{r^{1/n}} \cdot a_r$$

$$\underset{\to 1}{}$$

$\therefore$ By Sandwich, the reqd. limit exists and equals $a_r = \max\{a_1, a_2, \ldots, a_r\}$.

$$\lim_{n \to \infty} \left(\frac{a_1^n + a_2^n + \ldots + a_r^n}{n}\right)^{1/n} = \max\{a_1, \ldots, a_r\}.$$

5. $$n^2 < \underline{1^2 + 2^2 + \ldots + n^2} < \underline{n \times n^2}$$

$$\Rightarrow n^{2/n} < (1^2 + 2^2 + \ldots + n^2)^{1/n} < n^{3/n}$$

$\lim_{n \to \infty} n^{1/n} = 1$, so by Sandwich, reqd. limit is 1.

6. $$\frac{1}{n^2+n+1}+\frac{2}{n^2+n+2}+\dots+\frac{n}{n^2+n+n}$$

$$\sum_{k=1}^{n}\frac{k}{\underbrace{n^2+n+n}} \le \sum_{k=1}^{n}\frac{k}{n^2+n+k} \le \sum_{k=1}^{n}\frac{k}{\underbrace{n^2+n+1}}$$

$$= \frac{n(n+1)/2}{n(n+2)} \to 1/2 \qquad \frac{1}{2} \leftarrow = \frac{n(n+1)/2}{n^2+n+1}$$

By Sandwich,

the reqd. limit exists and equals $\frac{1}{2}$.

7. $$\frac{2n^2-7}{4n+5} < a_n < \frac{3n^2+8}{6n-1}$$

$$\frac{n(2n^2-7)}{(4n+5)(n+1)^2} < \frac{n\,a_n}{(n+1)^2} < \frac{n(3n^2+8)}{(n+1)^2(6n-1)}$$

$$= \frac{(2-7/n^2)}{(4+5/n)(1+1/n)^2} \qquad = \frac{(3+8/n^2)}{(1+1/n)^2(6-1/n)}$$

$$\to 2/4 = 1/2 \qquad \to 3/6 = 1/2$$

So, by Sandwich, the reqd limit exists, and equals 1/2.

8. (a) $x_n \to x \implies \sin x_n \to \sin x$.

$$|\sin x - \sin y| = \left|2\sin\frac{x-y}{2}\right|\left|\cos\frac{x+y}{2}\right|$$

$$\le 2\left|\sin\frac{x-y}{2}\right|$$

[For $t>0$, $\sin t < t$] $\leadsto$ $$\le 2\left|\frac{x-y}{2}\right| = |x-y|.$$

$|\sin x_n - \sin x| \le |x_n - x| \to$ Finish.

$\cos x_n = \sin\left(\frac{\pi}{2} - x_n\right) \to \sin\left(\frac{\pi}{2} - x\right) = \cos x.$

(b) $\sin x \le x \le \tan x$ for $x \in (0, \frac{\pi}{2})$.

$\sin \frac{1}{n} \le \frac{1}{n} \le \tan \frac{1}{n}$ $\left[\because \frac{1}{n} \in (0, \frac{\pi}{2})\right]$

$\Rightarrow \underbrace{\boxed{\cos \frac{1}{n}}}_{\text{conv. to } \cos 0, \text{ by part (a)}} \le n \sin \frac{1}{n} \le 1$

$\therefore$ By sandwich, $\underline{\lim_{n\to\infty} n \sin \frac{1}{n} = 1}$,

$$\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

(c) $\sin \frac{1}{n+1} < a_n < \sin \frac{1}{n}$

$\Rightarrow \underbrace{n \sin \frac{1}{n+1}}_{} < n a_n < \underbrace{\boxed{n \sin \frac{1}{n}}}_{\to 1 \text{ by part (b)}}$

$= (n+1) \sin \frac{1}{n+1} \times \frac{n}{n+1}$

$\to 1$ by part (b). So Sandwich ✓.
