Series of Real Numbers

$\{a_n\}_{n=1}^{\infty} \to$ sequence $\quad \underbrace{a_1 + a_2 + a_3 + (\ldots)}_{\text{Series}} \to ?$

$$S = 1 + (-1) + 1 + (-1) + (\ldots)$$
$$= (1+(-1)) + (1+(-1)) + \ldots$$
$$= 0 + 0 + 0 + \ldots = 0$$

$$S = 1 + (-1) + 1 + (-1) + (\ldots)$$
$$= 1 + ((-1)+1) + ((-1)+1) + \ldots$$
$$= 1 + 0 + 0 + \ldots = 1.$$

$$S = 1 + (-1) + 1 + (-1) + (\ldots)$$
$$= 1 - (1 + (-1) + 1 + (-1) + \ldots)$$
$$= 1 - S \Rightarrow S = \tfrac{1}{2}$$

$S = 0, 1,$ or $\tfrac{1}{2}$? $\boxed{\text{None}}$

OR, is it true that $0 = 1 = \tfrac{1}{2}$? X

Here, the $(\ldots)$ needs to be formally addressed.

How do we define $a_1 + a_2 + \ldots$?

$$\sum_{n=1}^{\infty} a_n \overset{\text{def}}{=} \lim_{N \to \infty} \sum_{n=1}^{N} a_n.$$

Define

$S_N = a_1 + a_2 + \ldots + a_N, \; N \geq 1.$ "partial sums"

We say that the series $\sum_{n \geq 1} a_n$ converges if this sequence of partial sums, $\{S_N\}_{N \geq 1}$ converges to a real number.

Example $\quad a_n = (-1)^n, \; n \geq 1.$

$$S_n = a_1 + a_2 + \ldots + a_n = \begin{cases} 0 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$$

$\therefore \{S_n\}_{n \geq 1}$ does not converge, hence $\sum_{n \geq 1} a_n$ does not have a meaning.

So, it does not make any sense to write

"Let $S = 1 + (-1) + 1 + (-1) + \ldots$"

Example

$$x_n = \frac{1}{n(n+1)}, \quad n \geq 1.$$

$$S_n = x_1 + x_2 + \dots + x_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

So, $\lim_{n\to\infty} S_n = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and equals 1.

Example $1 + r + r^2 + \dots = ?$

$$S_n = 1 + r + r^2 + \dots + r^{n-1} = \begin{cases} \frac{1-r^n}{1-r} & \text{if } r \neq 1 \\ n & \text{if } r = 1 \end{cases}$$

This converges only when $|r| < 1$.

So, $\sum_{n=0}^{\infty} r^n$ converges only when $|r| < 1$.

Also, when $|r| < 1$, $\sum_{n=1}^{\infty} r^{n-1} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1-r^N}{1-r} = \frac{1}{1-r}$.

We can write "let $S = a_1 + a_2 + \dots$" only when the series $\sum_{n=1}^{\infty} a_n$ converges.

Result

If $\{a_n\}_{n\geq 1}$ is a seq. of positive real numbers, $\sum_{n=1}^{\infty} a_n$ converges iff $\{S_N\}_{N\geq 1}$ is bounded. ($S_N = a_1 + \dots + a_N$)

Example $\sum_{n=1}^{\infty} 1/n^2$. $\rightarrow$ Positive terms, so enough to check whether it is bounded. $\sum_{n=1}^{N} 1/n^2 \leq \sum_{n=2}^{N} \frac{1}{(n-1)n} + 1 = 2 - \frac{1}{N} < 2$.

Hence, $\sum_{n=1}^{\infty} 1/n^2$ converges.

Result

If $\sum_{n\geq 1} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Why? $a_n = S_n - S_{n-1}$. If $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{N\to\infty} S_N = S$ (say)

Then $\lim_{N\to\infty} (S_N - S_{N-1}) = S - S = 0$.

Example $\sum_{n=1}^{\infty} r^{1/n}$; $r > 0$.

We know that $\lim_{n\to\infty} r^{1/n} = 1 \neq 0$, hence $\sum_{n=1}^{\infty} r^{1/n}$ does not converge.

Q: If $\lim_{n\to\infty} a_n = 0$, is it necessary that $\sum_{n=1}^{\infty} a_n$ converges?

Ans: No. e.g, $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

$\because H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ is unbounded.

$$\boxed{\begin{array}{c}\lim a_n = 0 \\ \not\Downarrow \quad \Uparrow \\ \sum a_n \text{ conv.}\end{array}}$$

Way 1: See note on Cauchy sequences.

Way 2: $$H_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\dots + \frac{1}{2^n}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^n} + \dots + \frac{1}{2^n}\right)$$
$$= 1 + \frac{n}{2}$$

This shows that $\{H_n\}_{n\geq 1}$ is not bounded.

Result $\sum_{n=1}^{\infty} (-1)^n a_n$ conv. if $\lim_{n\to\infty} a_n = 0$, and a_n's are positive.

proof: Exercise.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. (by the last result)

Cauchy criterion

$\sum_{n=1}^{\infty} a_n$ conv. iff $\{S_n\}_{n\geq 1}$ converges $\left[S_n = \sum_{k=1}^{n} a_k \right]$

iff $\underbrace{\{S_n\}_{n\geq 1} \text{ is Cauchy.}}_{\Updownarrow}$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $m > n \geq N$, it holds that $|S_m - S_n| < \varepsilon$, i.e.,

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon.$$

Example $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

First, $\left| \sum_{k=n+1}^{m} \frac{\sin k}{k^2} \right| \leq \sum_{k=n+1}^{m} \frac{|\sin k|}{k^2} \leq \sum_{k=n+1}^{m} \frac{1}{k^2}$.

Now, since $\sum \frac{1}{n^2}$ converges, we can say that for every $\varepsilon > 0$, $\exists N$ s.t. $\sum_{k=n+1}^{m} 1/k^2 < \varepsilon \ \forall m > n \geq N$.

Therefore, $\left| \sum_{k=n+1}^{m} \frac{\sin k}{k^2} \right| < \varepsilon$ for every $m > n \geq N$,

so by Cauchy criterion, $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ converges.

Result

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges too.

But the converse is not true.

proof: $\left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k|$, so the first part follows from Cauchy criterion.

Converse not true:

$\sum_{n\geq 1} a_n$ converges, but $\sum_{n\geq 1} |a_n|$ does not converge.

example: $\sum_{n\geq 1} \frac{(-1)^n}{n}$ converges, but $\sum_{n\geq 1} \frac{1}{n}$ does not converge.

Some names

When $\sum |a_n|$ converges, we say that $\sum a_n$ converges absolutely and when $\sum a_n$ converges but $\sum |a_n|$ does not, we say that $\sum a_n$ converges conditionally.

Comparison Test

If $|a_n| \leq b_n$ for every $n \geq 1$, where b_n is a positive seq.

(a) If $\sum_{n\geq 1} b_n$ converges then $\sum_{n\geq 1} a_n$ converges absolutely.

(b) If $\sum_{n\geq 1} a_n$ diverges then so does $\sum_{n\geq 1} b_n$.

Proof

(a) $S_N = \sum_{n=1}^{N} |a_n| \leq \sum_{n=1}^{N} b_n < \sum_{n=1}^{\infty} b_n$.

So, S_N converges ($\because$ inc & bdd) $\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

(b) Follows directly from part (a).

There are many others tests for convergence of a series. For example, Ratio Test, Root Test $\rightarrow$ not needed now. If interested, read from Bartle-Sherbert/Hunter's note given in website.

Rearrangement of a Series

$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ First, we find S.

$\boxed{\text{Fact}}$ $a_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \log n \longrightarrow$ converges. (exercise)

may require $\int \frac{dt}{t}$.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$$

$$= H_{2n} - H_n = (H_{2n} - \log 2n) - (H_n - \log n) + \log 2$$

$$= \underbrace{a_{2n} - a_n}_{\to 0} + \log 2$$

$$\therefore \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}\right) \to \log 2 \text{ as } n \to \infty.$$

$$\Rightarrow \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} + \frac{1}{2n+1}\right) \to \log 2 \text{ as } n \to \infty.$$

Thus, $\boxed{S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2}$ (Proved)

$$T = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \cdots$$

Q: We got T by rearranging the series $S = \log 2$. Will T be equal to $\log 2$?

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right)$$

$$= \left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{4n}\right)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n}\right)$$

$$- \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{4n}\right)$$

$$= H_{2n} - \frac{1}{2} H_n - \frac{1}{2} H_{2n}$$

$$= \frac{1}{2}\left(\underline{H_{2n} - H_n}\right)$$

[Earlier we showed that $H_{2n} - H_n \to \log 2$]

$$\to \frac{1}{2} \log 2 \quad \text{as } n \to \infty$$

$$T = \frac{1}{2} \log 2.$$

$$\boxed{\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots &= \log 2. \\ 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots &= \frac{1}{2} \log 2. \end{aligned}}$$

Is it Counter-intuitive? Update your intution here!

<u>Moral</u>: Value of an infinite series may change if we rearrange the series.

In fact, Riemann proved that any conditionally convergent series can be rearranged to converge to any given real no.

Q: Then when can we rearrange a series?

<u>Ans</u>: (i) If $\sum |a_n|$ conv. then you can rearrange $\sum a_n$.

(ii) If $\sum a_n$ is a series of positive terms only, then also you can rearrange the series. $(a_n > 0)$