Calculus : Day 2

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1. Subsequences

Suppose $\{x_n\}_{n\geq 1}$ is a sequence. A subsequence is just a sub-collection of the form $\{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\}$ where $1 \leq n_1 < n_2 < \cdots$. For example, if $x_n = 2n+1$, $n \geq 1$, then the sequence of odd primes: $\{3, 5, 7, 11, 13, \cdots\}$ is a subsequence of $\{x_n\}_{n\geq 1}$. [Although there is no formula for n_k's, they are well-defined: k^{th} odd prime $= 2n_k + 1$.]

Formally, we define subsequence as:

> **Definition** (Subsequence)
>
> $\{x_{n_k}\}_{k\geq 1}$ is called a subsequence of $\{x_n\}_{n\geq 1}$ if $n_k \in \mathbb{N}$ are such that $1 \leq n_1 < n_2 < n_3 < \cdots$.

Note, if $x_n = 2n+1$, $n \geq 1$, then $\{1, 3, 3, 5, 7, \cdots\}$ is not a subsequence of $\{x_n\}_{n\geq 1}$. Similarly, $\{3, 1, 5, 7, \cdots\}$ is not a subsequence of $\{x_n\}_{n\geq 1}$.

Subsequences enjoy more properties than the whole sequence has — lets explore some of them.

> 1. If $\{x_n\}_{n\geq 1}$ is bounded then any subsequence $\{x_{n_k}\}_{k\geq 1}$ is bounded as well.

Proof: $\{x_n\}_{n\geq 1}$ is bounded, so there exists $M > 0$ such that $|x_n| < M$ holds for all $n \in \mathbb{N}$. Hence, $|x_{n_k}| < M$ holds for all $k \geq 1$. Therefore, $\{x_{n_k}\}_{k\geq 1}$ is bounded. $\square$

> 2. If $\{x_n\}_{n\geq 1}$ is monotone, then so is any subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$.

Proof: You should be able to prove it yourself. $\square$

However, the converse of last theorem is not true: a sequence may have a monotone subsequence, but can be non-monotonic itself. For example, define

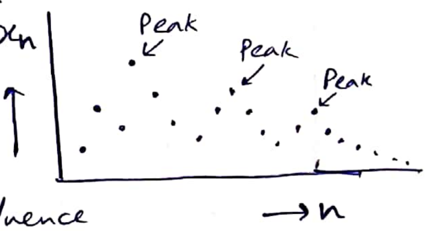
$$x_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} = (-1)^n, \; n \geq 1.$$

Then $\{x_{2n}\}_{n\geq 1}$ is monotonic, but $\{x_n\}_{n \geq 1}$ is not monotonic.

In fact, it turns out that every sequence has a monotonic subsequence:

$\boxed{\text{3. Any sequence has a monotonic subsequence.}}$

<u>Proof.</u> ~~We~~ Let $\{x_n\}_{n\geq 1}$ be any sequence. We say that x_n is a peak if $x_n \geq x_m$ for every $m \geq n$.



[Of course, it might happen that the sequence ~~do no~~ does not have a peak!] We have two cases —

<u>Case I</u>: There are infinitely many peaks. Then, we can form a 'sequence' of peaks, say $\{x_{n_1}, x_{n_2}, \ldots\}$, where $n_1 < n_2 < n_3 < \cdots$, which implies that

$$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots .$$

Therefore, this subsequence $\{x_{n_k}\}_{k \geq 1}$ is a decreasing subsequence, which proves the theorem in this case.

<u>Case II</u>: There are finitely many (may be zero) peaks, lets call them $x_{n_1}, x_{n_2}, \ldots, x_{n_m}$. Forget about the first n_m terms and look at $x_{n_m+1}, x_{n_m+2}, \ldots$. Let us call $n_m + 1 = l_1$. Since x_{l_1} is not a peak, so there exists $l_2 > l_1$ such that $x_{l_1} < x_{l_2}$. Again, since x_{l_2} is not a peak, so there exists $l_3 > l_2$ such that $x_{l_2} < x_{l_3}$. In this manner, we get

$\cancel{\ }$ $l_1 < l_2 < l_3 < \cdots$ such that $\cancel{x_{l_1}}$ $x_{l_1} > x_{l_2} > x_{l_3} > \cdots$.
Thus, we get an increasing sequence $\{x_{l_k}\}_{k\geq 1}$, which proves the theorem in this case. $\square$

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4. If $\{x_n\}_{n\geq 1}$ converges to x then any subsequence $\{x_{n_k}\}_{k\geq 1}$ must converge to x.

<u>Proof</u>: Fix any $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that, for every $n \geq N$, $x_n \in (x-\varepsilon, x+\varepsilon)$.

Now, $1 \leq n_1 < n_2 < n_3 < \cdots$, so there exists k_0 such that $n_{k_0} \geq N$. [In fact, $n_1 \geq 1, n_2 \geq 2, n_3 \geq 3, \ldots, n_N \geq N$.]

Then for every $k \geq k_0$, $n_k \geq n_{k_0} \geq N$, which implies that $x_{n_k} \in (x-\varepsilon, x+\varepsilon)$.

Thus, for every $\varepsilon > 0$, we get $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, $x_{n_k} \in (x-\varepsilon, x+\varepsilon)$. Hence, $\lim\limits_{k\to\infty} x_{n_k} = x$. $\square$

The last theorem is ~~very~~ intuitively easy, but has vast applications. In particular, we can use it to show that a sequence does not converge:

<u>Example</u>: Show that $\{(-1)^n\}_{n\geq 1}$ does not converge.

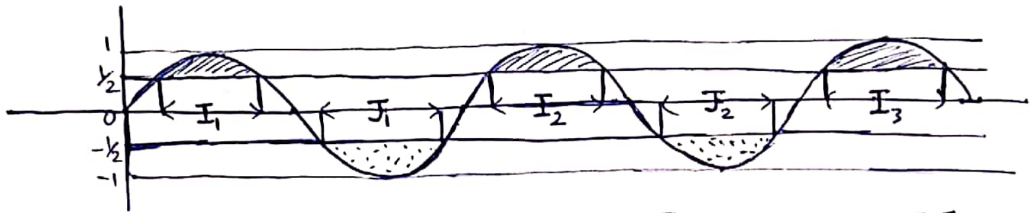
<u>Proof</u>: Let, if possible, $x_n = (-1)^n$ be convergent, say $\lim\limits_{n\to\infty} x_n = x$. Then, the subsequences $\{x_{2n}\}_{n\geq 1}$ and $\{x_{2n+1}\}_{n\geq 1}$ must also converge to x. But, $x_{2n} = (-1)^{2n} = 1$ for all $n \geq 1$ and $x_{2n+1} = (-1)^{2n+1} = -1$ for all $n \geq 1$.

So, $x_{2n} \to 1$ and $x_{2n+1} \to -1$ as $n \to \infty$. Thus, we get $x = 1 = -1$, contradiction.

Therefore, $x_n = (-1)^n$ does not converge. $\square$

Example Let $x_n = \sin n, n \geq 1$. Does $\{x_n\}_{n\geq 1}$ converge?

Solution Look at the graph of $f(x) = \sin x$, ~~xxx~~ $x \geq 0$.



Observe that $\sin x \geq \frac{1}{2}$ when $x \in \left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right]$, for some integer $n \geq 0$, and $\sin x \leq -\frac{1}{2}$ when $x \in \left[(2n+1)\pi + \frac{\pi}{6}, (2n+1)\pi + \frac{5\pi}{6}\right]$ for some integer $n \geq 0$. We define

$$I_{n+1} = \left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right], \ n \geq 0,$$

and $J_{n+1} = \left[(2n+1)\pi + \frac{\pi}{6}, (2n+1)\pi + \frac{5\pi}{6}\right]$, $n \geq 0$. $(n \in \mathbb{Z})$

Observe that for each $n \in \mathbb{N}$, I_n, J_n have length $\frac{2\pi}{3}$, which is bigger than 1. So, for each $n \in \mathbb{N}$, I_n, J_n each contain at least one integer. ~~Thus~~

Thus, for every $k \in \mathbb{N}$, we have an integer n_k that belongs to I_k and an integer m_k that belongs to J_k. Consider the two subsequences $\{\sin m_k\}_{k\geq 1}$, $\{\sin n_k\}_{k\geq 1}$.

If $\{\sin n\}_{n\geq 1}$ converges, say to l, then these two sub-sequences also converge to l.

Now, for each $k \in \mathbb{N}$, $\sin n_k \geq \frac{1}{2} \Rightarrow l = \lim_{k\to\infty} \sin n_k \geq \frac{1}{2}$.

(We are using exercise 1.24. in Day 1)

Similarly, for each $k \in \mathbb{N}$, $\sin m_k \leq -\frac{1}{2}$, which implies

$$l = \lim_{k\to\infty} \sin m_k \leq -\frac{1}{2}.$$

Thus, we get $\frac{1}{2} \leq l \leq -\frac{1}{2}$, which is absurd.

Therefore, $\{\sin n\}_{n\geq 1}$ does not converge. $\square$

Exercises

2.1. Define $x_n = \frac{n}{2} - \left[\frac{n}{2}\right]$, $n \geq 1$, where $[x]$ denotes the largest integer less than or equal to x. Does x_n converge?

2.2. Let $x_n = (1 - \frac{1}{n}) \sin \frac{n\pi}{2}$, $n \geq 1$. Show that $\{x_n\}_{n\geq 1}$ does not converge, but it has a convergent subsequence.

2.3. Suppose that $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are two convergent sequences, with $x = \lim\limits_{n\to\infty} x_n$ and $y = \lim\limits_{n\to\infty} y_n$. Define a new sequence $\{z_n\}_{n\geq 1}$ as —

$$z_{2n-1} = x_n \quad \text{and} \quad z_{2n} = y_n, \text{ for each } n \geq 1.$$

When does $\{z_n\}_{n\geq 1}$ converge? (Determine a necessary and sufficient condition for z_n to converge.)

2.4. Suppose that $\{x_n\}_{n\geq 1}$ is a sequence with the property that every convergent subsequence of $\{x_n\}_{n\geq 1}$ converges to same limit. Is it necessary that the whole sequence $\{x_n\}_{n\geq 1}$ ~~con~~ converges?

2.5. Suppose that $\{x_n\}_{n\geq 1}$ is a sequence such that every subsequence of it has a further subsequence (rather, sub subsequence) which converges. Is it necessary that $\{x_n\}_{n\geq 1}$ converges?

2.6. Suppose that $\{x_n\}_{n\geq 1}$ is a sequence such that every subsequence of it has a further subsequence which converges to same limit. Show that $\{x_n\}_{n\geq 1}$ must converge to that limit.

[Hint: What does it mean to say that "<u>x_n does not converges to x</u>"?]