Theorems on Continuity

Aditya Ghosh

Exercises/Problems

1. Let $f : I \to \mathbb{R}$ be a continuous function. If I is a closed bounded interval, then we know that f must be bounded. Show that the result fails to hold in each of the following cases: (a) I is bounded interval but not closed. (b) I is not bounded. (c) I is not an interval.

2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with the property that for every $x \in [a, b]$, there exists $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there exists $c \in [a, b]$ such that $f(c) = 0$.

3. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous and such that $f(a) < g(a)$ and $f(b) > g(b)$. Show that there exists $c \in (a, b)$ such that $f(c) = g(c)$.

4. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Let $x_1, x_2, \ldots, x_n$ be any n points in (a, b). Show that there exists $x_0 \in (a, b)$ such that
$$f(x_0) = \frac{1}{n}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right).$$

5. Prove that the equation $(1 - x)\cos x = \sin x$ has at least one solution in $(0, 1)$.

6. If $f : [0, 1] \to [0, 1]$ is a continuous function, then show that there exists $c \in [0, 1]$ such that $f(c) = c^2$.

7. Suppose that $f : [0, 1] \to [0, 1]$ is a continuous function with $f(0) = 0$ and $f(1) = 1$. Show that there exists $c \in (0, 1)$ such that $c^2 + (f(c))^2 = 1$.

8. Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous and $f(0) = f(2)$. Prove that there exists $a, b \in [0, 2]$ such that $b - a = 1$ and $f(b) = f(a)$.

9. An athlete runs a distance of 6 km in 30 minutes. Prove that somewhere during the run he covered a distance of 1 km in exactly 5 minutes.

10. Consider $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $f(x) = 1/x$. Is f continuous? Note that $f(-1) < 0 < f(1)$ and $f(x) \neq 0$ for any x. Does it contradict the intermediate value theorem?

11. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that there exists x_0 such that $f(x_0 + T/2) = f(x_0)$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that there exists x_0 such that $f(x_0 + \pi) = f(x_0)$. Convince yourself that the same result holds even if we replace π with any other number.

13. Suppose that f and g have the intermediate value property on some closed bounded interval I. Is it necessary that $f + g$ also has the intermediate value property on I?

14. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function that takes rational values only. What can you say about f?

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(q+1/n) = f(q)$ for every $q \in \mathbb{Q}$ and for every $n \in \mathbb{N}$. Show that f must be a constant function.

16. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and injective. Show that f must be strictly monotonic.

17. Let $f : [a, b] \to [c, d]$ be a strictly increasing function, where $c = f(a), d = f(b)$. Is it necessary that f^{-1} exists?

18. Let $f : [a, b] \to [c, d]$ be a continuous and strictly increasing function, where $c = f(a), d = f(b)$. Show that f^{-1} exists and is strictly increasing on the interval $[c, d]$. Furthermore, show that f^{-1} is continuous on $[c, d]$.

19. Suppose $x_1 = \tan^{-1} 2 > x_2 > x_3 > \cdots$ are positive real numbers, satisfying

$$\sin(x_{n+1} - x_n) + 2^{-(n+1)} \sin x_n \sin x_{n+1} = 0 \text{ for every } n \geq 1.$$

Find an expression for $\cot x_n$. Hence show that $\lim_{n\to\infty} x_n = \frac{\pi}{4}$.

20. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$be a polynomial with real coefficients, where $n > 0$ is even. If $a_n > 0$ and $a_0 < 0$, then show that the equation $f(x) = 0$ has at least two real roots.

21. Show that there exists a set of 100 consecutive integers of which exactly 19 are primes. (Hint: Do you know that there is a set of 100 consecutive integers which does not contain any prime?)

Solutions

1. (a) Take $I = (0, 1]$ and $f(x) = 1/x$. (b) Take $I = (0, \infty)$ and $f(x) = x$. (c) Take $I = [0, \pi/2) \cup (\pi/2, \pi]$ and $f(x) = \tan x$.

2. The function $|f|$ is continuous on $[a, b]$, hence attains a minimum value, say at c. But the given property with $x = c$ produces some y such that $|f(y)| \leq \frac{1}{2}|f(c)|$. Since $|f(c)| \leq |f(y)| \leq \frac{1}{2}|f(c)|$, it follows that $f(c) = 0$.

   Alternate solution: Start with some x_1. Let x_2 be the number such that $|f(x_2)| \leq \frac{1}{2}|f(x_1)|$. Next, x_3 be the number such that $|f(x_3)| \leq \frac{1}{2}|f(x_2)|$, and so on. Thus, we get a sequence x_n such that $|f(x_{n+1})| \leq \frac{1}{2}|f(x_n)|$ holds for every $n \geq 1$. It follows that $f(x_n) \to 0$ as $n \to \infty$. Now, x_n might not converge; but since it is bounded it has a convergent subsequence x_{n_k}. Suppose $x_{n_k} \to c$ as $k \to \infty$. Then $f(x_{n_k}) \to f(c)$ as $k \to \infty$. But earlier we had $f(x_{n_k}) \to 0$. Therefore, $f(c) = 0$.

3. Define $h(x) = f(x) - g(x)$. Since $h(a) < 0, h(b) > 0$ and h is continuous on $[a, b]$, the conclusion follows.

4. Let $m = \min\{f(x_1), f(x_2), \ldots, f(x_n)\}$ and $M = \max\{f(x_1), f(x_2), \ldots, f(x_n)\}$. Clearly, $m = f(x_i)$ and $M = f(x_j)$ for some $1 \leq i, j \leq n$. Now, the average $y = (f(x_1) + f(x_2) + \cdots + f(x_n))/n$ lies between $m = f(x_i)$ and $M = f(x_j)$. Therefore, there exists x_0 between x_i and x_j such that $f(x_0) = y$.

5. Consider $f(x) = (1 - x)\cos x - \sin x$. Observe that f is continuous, $f(0) > 0$ and $f(1) < 0$.

6. Consider $g(x) = f(x) - x^2$. Observe that g is continuous, $g(0) \geq 0$ and $g(1) \leq 0$.

7. Consider $g(x) = x^2 + (f(x))^2 - 1$.

8. Consider $g(x) = f(x + 1) - f(x)$. Observe that g is continuous, $g(0)$ and $g(1)$ are either zero, or have opposite signs. Thus, we get $c \in [0, 1]$ such that $g(c) = 0 \implies f(c + 1) = f(c)$. Set $b = c + 1$ and $a = c$.

9. Let x be the distance (in km) along the path. For $x \in [0, 5]$, let $f(x)$ denote the time elapsed (in minutes) for running from the point x to $x + 1$. The function f is continuous, and $f(0) + f(1) + \cdots + f(5) = 30$. Now use problem 4.

10. First, f is continuous on its domain. Second, it does not contradict the intermediate value theorem. Because in that theorem we assume that domain of f is an interval.

11. Consider $g(x) = f(x + T/2) - f(x)$. Observe that g is continuous, $g(0)$ and $g(T/2)$ are either zero, or have opposite signs. Thus, we get $x_0 \in [0, T/2]$ such that $g(x_0) = 0 \implies f(x_0 + T/2) = f(x_0)$.

12. Since $f : [0, T] \to \mathbb{R}$ is continuous, there exists $a, b \in [0, T]$ such that $f(a) \leq f(x) \leq f(b)$ holds for every $x \in [0, T]$. But f is periodic with period $T > 0$. Hence it follows that $f(a) \leq f(x) \leq f(b)$ holds for every $x \in \mathbb{R}$. Then we get $f(a) \leq f(a + \pi)$ and $f(b + \pi) \leq f(b)$. Therefore for $g(x) = f(x + \pi) - f(x)$, we have $g(a) \geq 0$ and $g(b) \leq 0$. The rest follows from the continuity of g.

13. No, $f + g$ need not have IVP. Here is a counter-example: Take $I = [-1, 1]$ and define

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad g(x) = \begin{cases} -\sin(1/x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

Show that f, g have IVP but $f + g$ does not have IVP.

14. We can show that f must be a constant function. Because if f takes at least two distinct values, say $a < b$, then the function has to take each value between a and b. But there is an irrational number between a and b, which can't belong to the range f.

15. Take a rational number $r = m/n$. If $m > 0$, then $f(\frac{m}{n}) = f(\frac{m-1}{n}) = \cdots = f(0)$. If $m < 0$, then $f(\frac{m}{n}) = f(\frac{m+1}{n}) = \cdots = f(0)$. Thus, for every $r \in \mathbb{Q}$, we have $f(r) = f(0)$. For any irrational x, take a sequence of rationals converging to x and use the continuity of f.

16. Fix any two numbers a, b, say $a < b$. Assume w.l.o.g. that $f(a) < f(b)$. We shall show that for any $x, y \in \mathbb{R}$, $x < y \iff f(x) < f(y)$. Take any x, y. Pick $M > 0$ sufficiently large such that $a, b, x, y \in [-M, M]$. Consider f on the interval $[-M, M]$. Since f is continuous and injective, f must be monotone on this interval. Since $-M \leq a < b \leq M$ and $f(a) < f(b)$, therefore f must be increasing on $[-M, M]$. Hence, $x < y \iff f(x) < f(y)$.

17. No. Consider $f : [0,2] \to [0,3]$ defined by $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ x+1 & \text{if } 1 \le x \le 2 \end{cases}$. However, the result holds if f is given to be continuous.

18. Take any $k \in (c,d)$. Since $f(a) < k < f(b)$ and f is continuous on $[a,b]$, there exists $c \in (a,b)$ such that $f(c) = k$. And since f is one-one, this c will be unique. Thus, f^{-1} exists and it is easy to show that it is strictly increasing. To show that f^{-1} is continuous, first take $y_0 = f(x_0)$. Our goal is to show that f^{-1} is continuous at y_0. Let me show the case $y_0 \in (c,d)$ and leave the boundary cases for you. Fix an $\varepsilon > 0$, small enough so that $x_0 - \varepsilon, x_0 + \varepsilon$ are within $[a,b]$. Since f is increasing, we have $f(x_0 - \varepsilon) < f(x) < f(x_0 + \varepsilon)$ for $x_0 - \varepsilon < x < x_0 + \varepsilon$. Take δ to be the minimum of the two positive numbers $f(x_0+\varepsilon) - f(x_0)$ and $f(x_0) - f(x_0 - \varepsilon)$. Note that for $y \in (y_0 - \delta, y_0 + \delta)$, we have $f(x_0 - \varepsilon) \le f(x_0) - \delta < y < f(x_0) + \delta \le f(x_0 + \varepsilon)$. (Draw a picture here!) Now, using the fact that f^{-1} is increasing, we get $x_0 - \varepsilon < f^{-1}(y) < x_0 + \varepsilon$. Thus, we have shown that for any (small enough) $\varepsilon > 0$, there exists $\delta > 0$ such that $|f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$ holds whenever $|y - y_0| < \delta$, which lets us to conclude that $f^{-1}(y)$ is continuous at $y = y_0$. (The boundary cases are similar: you just need to use one side instead of two.)

    Remark. Note that if we exclude the boundary points, then the above argument also applies when $c = -\infty$ or $d = \infty$. Here $c = f(a)$ should be taken in the sense that $c = \lim_{x \to a^+} f(x)$ (of course we have not discussed such limits yet, we will do that soon!), and similarly $d = \lim_{x \to b^-} f(x)$. For example, $f(x) = \tan(x)$ is a strictly increasing and continuous function from $(-\pi/2, \pi/2)$ to $(-\infty, \infty)$, and we know that $\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$ and $\lim_{x \to \frac{-\pi}{2}^+} \tan(x) = -\infty$. Therefore, its inverse $g(y) = \tan^{-1}(y)$ is continuous on $(-\infty, \infty)$.

    Corollary. Suppose that $f : I \to J$ is bijective and continuous, where $I, J \subseteq \mathbb{R}$ are <u>intervals</u>. Then f^{-1} is also continuous on its domain, J.

    (Since f is continuous and injective, it must be strictly monotone. Now consider two cases based on whether f is increasing or decreasing, and apply the above result.)

    Another remark. If the function is given to be continuous at just one point, then the version of the above result that we need is as follows: Suppose f is continuous at

x_0 and strictly monotone in an open neighborhood of x_0. Then we can say that f^{-1} will be continuous at $y_0 = f(x_0)$. (Do you see that the above proof goes smoothly even in this setup?)

19. First note that $\sin x_n \neq 0$ for each $n \geq 1$. Divide both sides by $\sin x_n \sin x_{n+1}$ and show that $\cot x_{n+1} - \cot x_n = \frac{1}{2^{n+1}}$ holds for every $n \geq 1$. Using this relation, along with the given fact that $\cot x_1 = 1/2$, deduce that for every $n \geq 1$,

$$\cot x_n = \cot x_{n-1} + \frac{1}{2^n} = \cot x_{n-2} + \frac{1}{2^{n-1}} + \frac{1}{2^n} = \cdots = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Hence, $\cot x_n = y_n \to 1$ as $n \to \infty$. Now, from the last problem we know that $\cot^{-1}$ is continuous at $y = 1$ (in fact, it is continuous on $\mathbb{R}$). Therefore, from $y_n \to 1$, we can say that $x_n = \cot^{-1}(y_n) \to \cot^{-1}(1) = \pi/4$.

Remark. Do you see why $\cot^{-1}$ is continuous on $\mathbb{R}$? To apply the result in the last problem, you need to take $(0, \pi)$ as the domain of $\cot(x)$, not $(-\pi/2, \pi/2)\setminus\{0\}$.

20. We proceed similar to what we did for showing that every polynomial of odd degree has at least one real root. First we observe/recall that

$$\lim_{x\to\infty} \frac{f(x)}{x^n} = a_n = \lim_{x\to-\infty} \frac{f(x)}{x^n}.$$

Since $a_n > 0$ and $n > 0$ is even, we can say that for some sufficiently large $a > 0$, the numbers $P(-a)$ and $P(a)$ are both positive. And we have $P(0) < 0$. Invoking the intermediate value property, we conclude that the polynomial has at least one zero in each of the intervals $(-a, 0)$ and $(0, a)$.

21. Observe that the set $S = \{1, 2, \ldots, 100\}$ contains 25 primes. And the set $T = \{101! + 2, 101! + 3, \ldots, 101! + 101\}$ does not contain any prime. Think of shifting the block of 100 numbers gradually so that we reach T from S. In each step, we include a new number and delete an old one. The number of primes in the block either increases by 1, or decreases by 1, or remains unchanged. Since it decreases to 0 from 25, there must be some stage when the block has exactly 19 primes. (Note that this argument is a discrete analog of the intermediate value property.)