A Tale of Rationals and Irrationals

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1. Show that between any two real numbers, there exists a rational number.

Solution. Take any $a, b \in \mathbb{R}$, say $a < b$. We wish to find $m, n \in \mathbb{Z}, n > 0$ such that $a < m/n < b$. Since $b - a > 0$, there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$ or, $nb > na + 1$. Then, we have at least one integer between na and nb, call that m. (In fact, $na < \lfloor na \rfloor + 1 \leq na + 1 < nb$ which tells us that we can take $m = \lfloor na \rfloor + 1$). Therefore, we have $na < m < nb \implies a < m/n < b$. □

2. Show that between any two real numbers, there exists an irrational number.

Solution. Take any $a, b \in \mathbb{R}$, say $a < b$. Using the above result, we have a rational number r such that $a - \sqrt{2} < r < b - \sqrt{2}$. Then, the number $r + \sqrt{2}$ is irrational and it lies between a and b. □

3. Show that for every $x \in \mathbb{R}$, there exists a sequence of rational numbers that converges to x.

Solution. Fix $x \in \mathbb{R}$. For every $n \in \mathbb{N}$, the interval $(x, x + 1/n)$ contains a rational number (by Problem 1 above), we call it q_n. Then, $x < q_n < x + 1/n$ holds for all $n \geq 1$. Sandwich theorem applies here and tells us that q_n converges to x. □

4. Show that for every $x \in \mathbb{R}$, there exists a sequence of irrational numbers that converges to x.

Solution. Fix $x \in \mathbb{R}$. For every $n \in \mathbb{N}$, the interval $(x, x+1/n)$ contains an irrational number (by Problem 2 above), we call it r_n. Then, $x < r_n < x + 1/n$ holds for all $n \geq 1$. Sandwich theorem applies here and tells us that r_n converges to x. □

5. Show that the following function is discontinuous everywhere

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Solution. Fix $x \in \mathbb{R}$. By Problems 3 and 4 above, there exists a sequence of rationals q_n and a sequence of irrationals r_n such that both of them converges to x. Observe that $f(q_n) = 0$ and $f(r_n) = 1$ for each $n \geq 1$. For f to be continuous at x, we must have $\lim_{n\to\infty} f(q_n) = f(x) = \lim_{n\to\infty} f(r_n)$ – which does not hold here. Therefore f is discontinuous at every $x \in \mathbb{R}$. □

6. Does there exist a function which is continuous only at $x = 0$?

Solution. Yes. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ x & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

For $a \neq 0$, we can show that f is not continuous at $x = a$ (argument is similar to the solution of the last problem). On the other hand, we can show that f is continuous at $x = 0$ as follows. Take any sequence x_n that converges to 0. Observe that $|f(x)| \leq |x|$ holds for all $x \in \mathbb{R}$. Therefore, $|f(x_n) - f(0)| = |f(x_n)| \leq |x_n| = |x_n - 0|$ holds for every $n \geq 1$. Using this, we get that $f(x_n) \to f(0)$ whenever $x_n \to 0$. □

7. Does there exist a function defined on $\mathbb{R}$ which is discontinuous everywhere except at the integers? (i.e. f is continuous at $x = n$ if and only if $n \in \mathbb{Z}$.)

Solution. Yes. Consider the function

$$f(x) = \begin{cases} \sin \pi x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

First, for $a \notin \mathbb{Z}$ we show that f is not continuous at $x = a$. Take a sequence of rationals r_n and a sequence of irrationals s_n, both of which converge to a. We have $f(r_n) = \sin \pi r_n$ and $f(s_n) = 0$ for $n \geq 1$, and $\sin \pi r_n \to \sin \pi a$ (since $t \mapsto \sin t$ is continuous). Now for f to be continuous at $x = a$, we must have $\lim\limits_{n\to\infty} f(r_n) = \lim\limits_{n\to\infty} f(s_n)$, implying that $\sin \pi a = 0$ which is not possible since $a \notin \mathbb{Z}$.

Next, fix $a \in \mathbb{Z}$. Let x_n be any sequence that converges to a. Note that if x_n is rational, we have $|f(x_n) - f(a)| = |\sin \pi x_n - \sin \pi a| \leq \pi |x_n - a|$. And if x_n is irrational then $|f(x_n) - f(a)| = 0$. *In either case,* we have $|f(x_n) - f(a)| \leq \pi |x_n - a|$ for every $n \geq 1$. Using this, we conclude that $f(x_n) \to f(a)$ whenever $x_n \to a$. □

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(r) = cr$ for all $r \in \mathbb{Q}$, where $c = f(1)$. Can you say anything more?

Solution. First observe that $f(n) = nf(1) = cn$ for all $n \in \mathbb{N}$. Next, observe that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Using these, we get $f(n) = cn$ for all $n \in \mathbb{Z}$. Next, observe that for each $k \in \mathbb{N}$, we have $f(kx) = kf(x)$ for all $x \in \mathbb{R}$. Now, take any rational number $r = m/n$ where $m, n \in \mathbb{Z}, n > 0$. We have $cm = f(m) = f(nr) = nf(r) \implies f(r) = cr$. This completes the proof. Without further assumptions, it is hard to tell anything more for f. $\square$

9. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = cx$ for all $r \in \mathbb{R}$, where $c = f(1)$.

Solution. We continue from the previous solution. We derived that $f(r) = cr$ for every $r \in \mathbb{Q}$. Now, take any $x \in \mathbb{R}$. There exists a sequence of rationals q_n that converges to x. We have $f(q_n) = cq_n$ for each $n \geq 1$. Letting $n \to \infty$ and using the continuity of f at x, we get $f(x) = \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} cq_n = cx$. Since this holds for every $x \in \mathbb{R}$, we are done. $\square$

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$ and satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = cx$ for all $r \in \mathbb{R}$, where $c = f(1)$. (Note, f is *given* to be continuous only at $x = 0$. It does not mean that f is discontinuous at other points.)

Solution. We have $f(r) = cr$ for every $r \in \mathbb{Q}$. Now, fix any $x \in \mathbb{R}$. There exists a sequence of rationals q_n that converges to x. We have $f(q_n) = cq_n$ for each $n \geq 1$. Consider the sequence $r_n = q_n - x$. Since $r_n \to 0$ as $n \to \infty$ and f is continuous at $x = 0$, we obtain that $f(r_n) \to f(0) = 0$ as $n \to \infty$. Now, observe that $f(x) = f(q_n) - f(q_n - x) = cq_n - f(r_n)$ for every $n \geq 1$. Letting $n \to \infty$ here, we get

$$f(x) = \lim_{n\to\infty} (cq_n - f(r_n)) = \lim_{n\to\infty} cq_n - \lim_{n\to\infty} f(r_n) = cx - 0 = cx.$$

This completes the proof. $\square$

Alternate Solution: Fix any $x \in \mathbb{R}$ and take any sequence x_n that converges to x. Then, $y_n = x_n - x$ converges to 0, hence $f(y_n)$ converges to $f(0) = 0$. Therefore, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(y_n) - 0| < \varepsilon$ holds for all $n \geq N$. Then, we have $|f(x_n) - f(x)| = |f(x_n - x)| = |f(y_n)| < \varepsilon$ for every $n \geq N$. Hence $f(x_n)$ converges to $f(x)$. Since x is arbitrary, we conclude that f is continuous everywhere and hence the result in Problem 9 applies here. $\square$

Exercises

1. Does there exist a function defined on $\mathbb{R}$ which is discontinuous everywhere except at $x = 0, 1, 2, \ldots, 10$?

2. Define $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1 - x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Discuss the continuity of f.

3. Let f and g be continuous on the interval $[a, b]$. Suppose that $f(r) = g(r)$ for all rational $r \in [a, b]$. Is it necessary that $f(x) = g(x)$ for every $x \in [a, b]$?

4. Suppose $f : (0, \infty) \to \mathbb{R}$ satisfies $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Furthermore, assume that f is continuous on $(0, \infty)$. Find all such functions f.

5. Determine all continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ that satisfy the following relation $f(x + y) = g(x) + h(y)$ for all $x, y \in \mathbb{R}$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and suppose that $f(x)$ is continuous at $x = 0$.

   (a) Show that $f(0) = 0$ or 1. What happens if $f(0) = 0$?

   (b) If $f(0) \neq 0$ then show that $f(x) \neq 0$ for all $x \in \mathbb{R}$. We shall assume $f(0) \neq 0$ in the subsequent parts as well.

   (c) Show that f is continuous at every point.

   (d) Call $f(1) = c$. Find $f(2), f(-1), f(1/2)$ in terms of c. Can you find $f(r)$ for any $r \in \mathbb{Q}$?

   (e) Can you find $f(\sqrt{2})$ in terms of c? Can you find $f(x)$ for any $x \in \mathbb{R}$?

7. Let $g, h : \mathbb{R} \to \mathbb{R}$ be continuous functions. Define $f(x) = \begin{cases} g(x) & \text{if } x \text{ is rational,} \\ h(x) & \text{if } x \text{ is irrational.} \end{cases}$

   Show that $f(x)$ is continuous at exactly those points where g and h are equal.

8. Consider the following function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/q & \text{if } x = p/q \text{ where } p \in \mathbb{Z}, q \in \mathbb{N} \text{ and } p, q \text{ are coprime.} \end{cases}$$

(a) Show that $f(x+n) = f(x)$ holds for every $x \in \mathbb{R}, n \in \mathbb{Z}$. This says that we can concentrate on the behavior of f in $[0, 1]$ only.

(b) Show that f is discontinuous at all rational numbers.

(c) Show that f is continuous at all irrational numbers. In view of (a), it is enough to prove this for the irrationals in $(0, 1)$ only.

Hints/Answers

1. Yes. One example is the following function: $f(x) = 0$ if x is irrational and $x(x-1)\cdots(x-10)$ if x is rational.
2. f is discontinuous everywhere except at $x = 1/2$.
3. Fix $x \in [a, b]$. Take a sequence of rationals that converges to x.
4. Put $x = e^a, y = e^b$ and use problem 9 above. Use the function $g(x) = f(e^x)$.
5. Show that $g(x) - g(0) = h(x) - h(0)$ for all x. Call $g(x) - g(0) = k(x)$. Since $f(0) = g(0) + h(0)$, we obtain $f(x+y) - f(0) = k(x) + k(y)$ for all x, y. Now show that $k(x) = f(x) - f(0)$. Hence we arrive at $k(x+y) = k(x) + k(y)$ for all x, y. Now use problem 9 above.
6. Carry out the steps given. You will get that $f(x) = c^x$ for all x, where $c = f(1)$.
7. If $g(a) \neq h(a)$ it is easy to show that f is discontinuous at $x = a$ (take two sequences). If $g(a) = h(a)$, you can use the following bound: $|f(x) - f(a)| \leq \max\{|g(x) - g(a)|, |h(x) - h(a)|\}$.
8. Furhter hint for part (c) : There are only finitely many rational numbers in $[0, 1]$ whose denominator, in reduced form, is less than any fixed number N. So if we pick N such that $1/N < \varepsilon$, then for any sequence x_n converging to an irrational number r, only finitely many $f(x_n)$'s can be bigger than ε.

   The function given here is known as Thomae's function. You can search in Wikipedia to know more about this function.