① $\frac{1}{4}\sum_{n=1}^{N} \frac{n+4-n}{n(n+2)(n+4)} = \frac{1}{4}\sum_{n=1}^{N}\left[\frac{1}{n(n+2)} - \frac{1}{(n+2)(n+4)}\right]$

$= \frac{1}{4}\left[\frac{1}{1(1+2)} + \frac{1}{2(2+2)} - \frac{1}{(N+1)(N+3)} - \frac{1}{(N+2)(N+4)}\right]$

$\longrightarrow \frac{1}{4}\left(\frac{1}{3}+\frac{1}{8}\right) = \frac{11}{96}$ as $N \to \infty$.

② $\frac{(n+1)}{(n-1)! + n! + (n+1)!} = \frac{\cancel{(n+1)}}{\cancel{(n+1)}\left[(n-1)! \times (n+1)\right]} = \frac{n}{(n+1)!}$

$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{n+1-1}{(n+1)!} = \sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = 1.$

$\sum_{n=1}^{N} \frac{1}{\sqrt{n}\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})} = \sum_{n=1}^{N}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{N+1}} \to 1.$

③ (a) Since the terms are positive, we can rearrange!

$1 + \frac{1}{2} + \frac{1}{4^2} + \frac{1}{2^3} + \frac{1}{4^4} + \frac{1}{2^5} + \frac{1}{4^6} + \cdots$

$= \left(1 + \frac{1}{4^2} + \frac{1}{4^4} + \cdots\right) + \left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots\right)$

$= \frac{1}{1 - 1/4^2} + \frac{1/2}{1 - 1/2^2} = \frac{26}{15}.$

(b) $1 + \left(\frac{1}{2} + \frac{1}{2\times 3}\right) + \left(\frac{1}{2^2\times 3} + \frac{1}{2^2\times 3^2}\right) + \cdots$

$= 1 + \frac{4}{2\times 3} + \frac{4}{2^2\times 3^2} + \frac{4}{2^3\times 3^3} + \cdots = 1 + 4\,\frac{1/6}{1 - 1/6} = \frac{9}{5}.$

④ $\sum_{n=1}^{N} \frac{1}{a_n a_{n+2}} = \sum_{n=1}^{N} \frac{a_{n+1}}{a_n a_{n+1} a_{n+2}}$

$a_{n+2} = a_n + a_{n+1}$, $a_1 = 1$, $a_2 = 2$.

Prove that $a_n \geq n$ (by induction).

$= \sum_{n=1}^{N} \frac{a_{n+2} - a_n}{a_n a_{n+1} a_{n+2}} = \sum_{n=1}^{N}\left(\frac{1}{a_n a_{n+1}} - \frac{1}{a_{n+1} a_{n+2}}\right)$

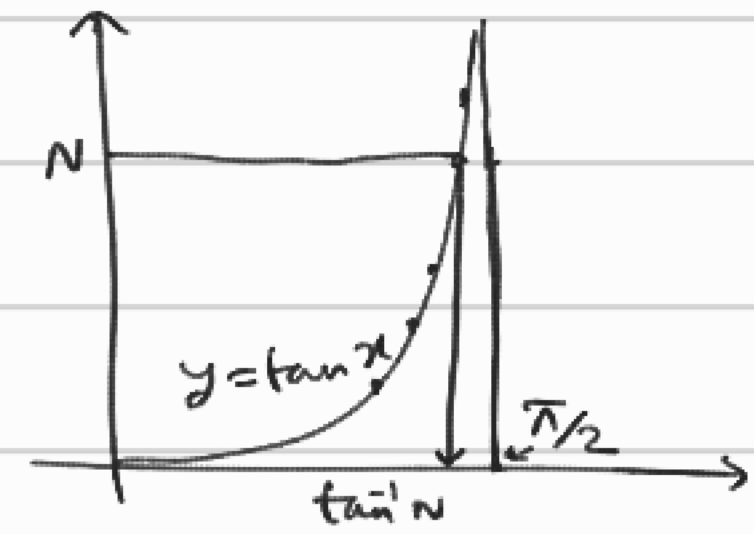
$= \frac{1}{a_1 a_2} - \frac{1}{a_{N+1} a_{N+2}} \longrightarrow \frac{1}{a_1 a_2}$

$\left[\because\ 0 < \frac{1}{a_{n+1} a_{n+2}} \leq \frac{1}{(n+1)(n+2)} \to 0.\right]$

⑤ $$\sum_{n=1}^{N} \tan^{-1}\frac{n+1-n}{1+n(n+1)} = \sum_{n=1}^{N}\left(\tan^{-1}(n+1) - \tan^{-1}(n)\right)$$

$$= \tan^{-1}(N+1) - \tan^{-1}(1)$$

$$\left[\lim_{N\to\infty} \tan^{-1}(N) = \frac{\pi}{2}\right] \longrightarrow \frac{\pi}{2} - \frac{\pi}{4} \text{ as } N\to\infty.$$



⑥ $$\sum \sin\left(\pi\sqrt{n^2+n+1}\right)$$

$[\sqrt{n^2+n+1}] = n$ $\quad$ $\sin(x) = (-1)^n \sin(x-n\pi)$

$$= \sum (-1)^n \sin\left(\pi\left(\sqrt{n^2+n+1}-n\right)\right)$$

$$= \sum (-1)^n \sin\left(\pi\frac{n+1}{\sqrt{n^2+n+1}+n}\right)$$

$$\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-n\right) = \lim_{n\to\infty}\frac{n+1}{\sqrt{n^2+n+1}+n} = \frac{1}{2}.$$

(by an earlier exc, $x_n\to x \Rightarrow \sin x_n \to \sin x$) $\to \sin\frac{\pi}{2} = 1$

Let, if possible, the series converge. Then we can say that

$$\lim_{n\to\infty} (-1)^n \sin\left(\pi\left(\sqrt{n^2+n+1}-n\right)\right) = 0$$

$$\Rightarrow \lim_{n\to\infty}\left|\sin\left(\pi\left(\sqrt{n^2+n+1}-n\right)\right)\right| = 0.$$ But $$\lim_{n\to\infty}\left|\sin\left(\pi\left(\sqrt{n^2+n+1}-n\right)\right)\right| = \sin\frac{\pi}{2} = 1.$$ Contradiction.

⑦ $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n}$ converges. $\qquad 9\times 10^{k-1} - 9^{k-1}$

n does not contain zero

$a_n = n^{th}$ natural no. that does not contain a zero.

These are the numbers that consist of digits 1 to 9 only.

How many k-digited no.s can you form with digits 1 to 9? Ans: 9^k.

Min such number is $\underbrace{11\cdots1}_{k \text{ ones}} > 10^{k-1}$

$$\left(\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{9}\right) + \left(\frac{1}{11}+\frac{1}{12}+\dots+\frac{1}{19}+\frac{1}{21}+\frac{1}{22}+\dots+\frac{1}{99}\right) + \left(\frac{1}{111}+\frac{1}{112}+\dots\right)+\dots$$

So, $$\sum_{\text{no. of digits}=k}\frac{1}{a_n} \le 9^k\times\frac{1}{10^{k-1}}$$

$$\sum_{n=1}^{\infty}\frac{1}{a_n} = \sum_{k=1}^{\infty}\left(\sum_{\text{no. of digits}=k}\frac{1}{a_n}\right) \le \sum_{k=1}^{\infty} 9\left(\frac{9}{10}\right)^{k-1} = 90.$$

(8) $x^2-1=(x-1)(x+1) \Rightarrow \dfrac{2}{x^2-1} = \dfrac{(x+1)-(x-1)}{(x-1)(x+1)} = \dfrac{1}{x-1}-\dfrac{1}{x+1}$.

$$\underbrace{\frac{1}{1-x}+\frac{1}{x+1}}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+\dots+\frac{2^{n-1}}{1+x^{2^{n-1}}}$$

$$=\underbrace{\frac{2}{1-x^2}+\frac{2}{1+x^2}}+\frac{4}{x^4+1}+\dots+\frac{2^{n-1}}{1+x^{2^{n-1}}}$$

$$=\frac{4}{1-x^4}+\frac{4}{1+x^4}+\dots+\frac{2^{n-1}}{1+x^{2^{n-1}}}$$

$$\vdots$$

$$=\frac{2^n}{1-x^{2^n}}.$$

So, $\dfrac{1}{x+1}+\dfrac{2}{x^2+1}+\dfrac{4}{x^4+1}+\dots+\dfrac{2^{n-1}}{1+x^{2^{n-1}}}$

$$=\frac{1}{x-1}-\frac{2^n}{x^{2^n}-1}.$$

Enough to show that $\lim_{n\to\infty} 2^n/(x^{2^n}-1)=0$.

Let us fix x,
$r=|x|>1$.

$$\frac{2^n}{x^{2^n}-1}=\frac{2^n}{|x|^{2^n}-1}=\frac{2^n}{r^{2^n}-1}\to \text{subseq. of } \frac{n}{r^n-1}.$$

Note that it suffices to show that for $r>1$,

$$\lim_{n\to\infty}\frac{n}{r^n-1}=0.$$

$r>1$
Let $r=1+\delta$.

$$0<\frac{n}{(1+\delta)^n-1}<\frac{n}{\binom{n}{2}\delta^2} \quad \left[\because (1+\delta)^n=\sum_{k=0}^{n}\binom{n}{k}\delta^k>1+\binom{n}{2}\delta^2\right]$$

$\therefore$ By Sandwich, $\lim_{n\to\infty}\dfrac{n}{r^n-1}=0$.

(9) $\sum_{n=1}^{\infty}|x_{n+1}-x_n|$ converges $\Rightarrow \sum_{n=1}^{\infty}(x_{n+1}-x_n)$ converges

$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}(x_{k+1}-x_k)$ exists $\Rightarrow \lim_{n\to\infty}(x_{n+1}-x_1)$ exists

$\Rightarrow \lim_{n\to\infty} x_n$ exists.

(10) $\sum_{n=1}^{N}\frac{1}{n^a}\geq\sum_{n=1}^{N}\frac{1}{n}$ ($\because a\leq 1$) and $\underbrace{\sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges}}_{\text{proved earlier}}$, so $\sum_{n=1}^{\infty}\frac{1}{n^a}$ must also diverge for $a\leq 1$.

(11) (Alt. proof of "$\sum_{n\geq 1} |a_n|$ converges $\Rightarrow \sum_{n\geq 1} a_n$ converges")

$0 \leq a_n + |a_n| \leq 2|a_n|$.

$\therefore$ If $\sum_{n\geq 1} |a_n|$ converges, then $\sum_{n\geq 1} (a_n + |a_n|)$ also converges,

so $\sum_{n\geq 1} a_n$ must also converge.

(12) (a) easy. $x = x^+ - x^-$, $|x| = x^+ + x^-$

$x^+, x^- \geq 0$

(b) $|a_n| \geq a_n^+, a_n^- \geq 0$, so $\sum |a_n|$ conv $\Rightarrow \sum a_n^+, \sum a_n^-$ converges

$\sum a_n^+$ and $\sum a_n^-$ conv $\Rightarrow \sum (a_n^+ + a_n^-)$ conv $\Rightarrow \sum |a_n|$ conv.

(13) $\sum (-1)^{n-1} a_n$ where $a_n \downarrow 0$. Define $S_n = \sum_{k=1}^{n} (-1)^{k-1} a_k$, $n \geq 1$.

$$S_{2n} = \underbrace{(a_1 - a_2)}_{\geq 0} + \underbrace{(a_3 - a_4)}_{\geq 0} + \underbrace{(a_5 - a_6)}_{\geq 0} + \dots + \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0}$$

$\therefore S_{2n} \geq 0$ and increasing.

Also, $S_{2n} = a_1 + \underbrace{(-a_2 + a_3)}_{\leq 0} + \dots + \underbrace{(-a_{2n-2} + a_{2n-1})}_{\leq 0} - a_{2n}$

$< a_1 - a_{2n} < a_1$.

$\therefore S_{2n}$ inc and bdd above $\Rightarrow \lim_{n\to\infty} S_{2n}$ exists.

$S_{2n+1} = S_{2n} + \underbrace{a_{2n+1}}_{\to 0}$ $\therefore \lim_{n\to\infty} S_{2n+1}$ also exists and equals $\lim_{n\to\infty} S_{2n}$.

(14) $a_n \downarrow 0$, $\sum_{n\geq 1} a_n$ converges. Define $S_n = \sum_{k=1}^{n} a_k$.

$\Downarrow$

$\lim_{n\to\infty} S_n$ exists, say S.

$S_{2n} - S_n = a_{n+1} + a_{n+2} + \dots + a_{2n} \geq n\, a_{2n}$ $\qquad n a_n$

$\Rightarrow 0 < 2n\, a_{2n} < \underbrace{2(S_{2n} - S_n)}_{(\text{as } n\to\infty) \to 2(S-S) = 0}$ $\therefore$ By Sandwich,

$\lim_{n\to\infty} 2n\, a_{2n} = 0.$

Also, $0 \leq (2n+1)\, a_{2n+1} \leq \boxed{\frac{2n+1}{2n}} \boxed{2n\, a_{2n}} \xrightarrow{\text{Sandwich}} \lim_{n\to\infty} (2n+1)\, a_{2n+1} = 0.$

DONE

$\sum_{n\geq 1} a_n$ Conv $\overset{?}{\Rightarrow}$ $\lim n a_n = 0$ Ans: No. HW Find counterexample.

(15) $a_{n+1} = \dfrac{n a_n + a_{n-1}}{n+1} \Rightarrow a_{n+1} - a_n = -\dfrac{a_n - a_{n-1}}{n+1}$

$$a_{n+1} - a_n = -\frac{a_n - a_{n-1}}{n+1} = +\frac{a_{n-1} - a_{n-2}}{(n+1)n} = \cdots = (-1)^{n-1}\frac{(a_2 - a_1)2}{(n+1)!}$$

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| \leq 2|a_2 - a_1| \sum_{n=1}^{\infty} \frac{1}{(n+1)!} < \infty.$$

$\Rightarrow \lim_{n\to\infty} a_n$ exists (by an exercise above).

How to find the limit?

$$a_{n+1} - a_n = (-1)^{n-1}\frac{2(a_2 - a_1)}{(n+1)!}$$

$$a_{n+1} - a_1 = \sum_{k=1}^{n} (a_{k+1} - a_k) = 2(a_2 - a_1) \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(k+1)!}$$

$$\to 2(a_2 - a_1)\left(\underbrace{1 - 1/1!} + 1/2! - 1/3! + \cdots\right) = \frac{2}{e}(a_2 - a_1)$$

$\lim_{n\to\infty} a_n = a_1 + \frac{2}{e}(a_2 - a_1)$. $\left[e^x = \sum_{k=0}^{\infty} x^k/k! \text{ for all } x \in \mathbb{R}.\right]$
(We'll see later)

(16) $\sum_{n=1}^{\infty} a_n$ Converges and $a_n \geq 0$

$$\sum_{n=1}^{N} \underbrace{\sqrt{a_n a_{n+1}}}_{\geq 0} \leq \sum_{n=1}^{N} \frac{a_n + a_{n+1}}{2} \leq \sum_{n=1}^{\infty} a_n < \infty.$$

(17) $a_{n+1} - 1 = \dfrac{a_n^2 - 1}{2} \Rightarrow \dfrac{1}{a_{n+1} - 1} = \dfrac{2}{(a_n - 1)(a_n + 1)}$

$$\Rightarrow \frac{1}{a_{n+1} - 1} = \frac{1}{a_n - 1} - \frac{1}{a_n + 1}$$

$$\Rightarrow \frac{1}{a_n + 1} = \frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1}, \text{ for all } n \geq 1.$$

$$\sum_{k=1}^{n} \frac{1}{a_k + 1} = \sum_{k=1}^{n} \left(\frac{1}{a_k - 1} - \frac{1}{a_{k+1} - 1}\right) = \frac{1}{a_1 - 1} - \frac{1}{a_{n+1} - 1} = \frac{1}{2} - \frac{1}{a_{n+1} - 1}.$$

This proves the given identity. Next, $\sum_{k=1}^{\infty} \frac{1}{a_k + 1} = \frac{1}{2} - \underbrace{\lim_{n\to\infty} \frac{1}{a_n - 1}}_{?}$.

$a_1 = 3, \quad a_{n+1} = \dfrac{a_n^2+1}{2}.$ Show $a_n \geq n$ by induction.

Then $\lim\limits_{n\to\infty} \dfrac{1}{a_n - 1} = 0 \Rightarrow \sum\limits_{n=1}^{\infty} \dfrac{1}{a_n+1} = \dfrac{1}{2}.$

(18) $\langle n \rangle$ = integer closest to $\sqrt{n}$ $= [\sqrt{n}+\frac{1}{2}]$ (prove)

$[\cdot\cdot] \to$ floor

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$

Find all $n \in \mathbb{N}$ s.t. $\langle n \rangle = k$.

$$[\sqrt{n}+\tfrac{1}{2}] = k$$

$$\Leftrightarrow \quad k \leq \sqrt{n} + \tfrac{1}{2} < k+1$$

$$\Leftrightarrow \quad (k-\tfrac{1}{2})^2 \leq n < (k+\tfrac{1}{2})^2$$

$$\Leftrightarrow \quad k^2 - k + \tfrac{1}{4} \leq n < k^2 + k + 1/4$$

$$\Leftrightarrow \quad \underline{k^2 - k + 1 \leq n \leq k^2 + k}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n:\langle n \rangle = k} \frac{2^k+2^{-k}}{2^n}\right)$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n=k^2-k+1}^{k^2+k} \frac{2^k+2^{-k}}{2^n}\right)$$

$$= \sum_{k=1}^{\infty}\left((2^k+2^{-k})\frac{1}{2^{k^2-k+1}} \frac{\left(1-\frac{1}{2^{2k}}\right)}{1-\frac{1}{2}}\right)$$

$$= \sum_{k=1}^{\infty} (2^k+2^{-k}) \frac{1}{2^{k^2-k}} \times \frac{(2^k-2^{-k})}{2^k}$$

$$= \sum_{k=1}^{\infty} \frac{2^{2k}-2^{-2k}}{2^{k^2}} = \sum_{k=1}^{\infty}\left(\frac{1}{2^{k^2-2k}} - \frac{1}{2^{k^2+2k}}\right) = 2\sum_{k=1}^{\infty}\left(\frac{1}{2^{(k-1)^2}} - \frac{1}{2^{(k+1)^2}}\right)$$

$$= 2\left(\frac{1}{2^0}+\frac{1}{2^1}\right) = 3. \quad \text{(Ans)}$$

(19) $x_{n+1} = \left(1-\frac{1}{2n}\right)x_n + \frac{1}{2n}x_{n-1}, \quad n \geq 1.$ $\quad x_0 = a,$ $x_1 = b.$

$$(x_{n+1}-x_n) = -\frac{1}{2n}(x_n - x_{n-1}).$$

$$(x_{n+1}-x_n) = \frac{(-1)^n (x_1-x_0)}{2^n\, n!}$$

$$x_n - x_0 = \sum_{k=1}^{n}(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{(-1)^{k-1}(x_1-x_0)}{2^{k-1}(k-1)!}$$

$$\left[\sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x\right]$$

$$\Rightarrow \lim_{n\to\infty} x_n = x_0 + (x_1-x_0)\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k\, k!} = x_0 + \frac{(x_1-x_0)}{\sqrt{e}}$$

(20) (a proof of $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$)

(a) $(\cos\theta + i\sin\theta)^{2n+1} = \underbrace{\cos(2n+1)\theta}_{(-1)^k} + i\,\underbrace{\sin(2n+1)\theta}_{=0}$ $\quad \left[\theta = \frac{k\pi}{2n+1}\right]$

$\Rightarrow \sum_{k=0}^{2n+1} \binom{2n+1}{k} (\cos\theta)^{2n+1-k} (i\sin\theta)^k = (-1)^k + i \times 0 \rightarrow$ real

$\therefore$ Imaginary part $= 0 \Rightarrow \sum_{\ell=0}^{n} \binom{2n+1}{2\ell+1} (\cos\theta)^{2n+1-(2\ell+1)} (i\sin\theta)^{2\ell+1} = 0$

$\left[\text{divide by } (\sin\theta)^{2n+1}\right] \Rightarrow \sum_{\ell=0}^{n} (-1)^\ell \binom{2n+1}{2\ell+1} (\cot^2\theta)^{n-\ell} = 0$ $\quad \left[\because \sin\frac{k\pi}{2n+1} \neq 0\right]$ $(k = 1, \dots, n)$

$\boxed{F(x) = \sum_{\ell=0}^{n} (-1)^\ell \binom{2n+1}{2\ell+1} x^{n-\ell}}$ What we have shown above is that,

$F(\cot^2\theta) = 0$ for $\theta = \frac{k\pi}{2n+1}$, $k = 1, 2, \dots, n$.

(b) Degree of $F(x)$ is n. So F can not have more than n distinct zeros. And $\left\{\cot^2\frac{k\pi}{2n+1}, k = 1, \dots, n\right\}$ are n distinct roots, so they are the only roots of $F(x) = 0$. Therefore, by Vieta's theorem,

$$\sum_{k=1}^{n} \cot^2\frac{k\pi}{2n+1} = -\frac{(-1)^1\binom{2n+1}{3}}{(-1)^0\binom{2n+1}{1}} = \frac{n(2n-1)}{3}.$$

(c) $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$. $\quad \frac{k\pi}{2n+1} \in \left(0, \frac{\pi}{2}\right)$ for $k = 1, \dots, n$.

$\Rightarrow \operatorname{cosec}^2 x > \frac{1}{x^2} > \cot^2 x$

$\Rightarrow \cot^2\frac{k\pi}{2n+1} + 1 > \frac{(2n+1)^2}{k^2\pi^2} > \cot^2\frac{k\pi}{2n+1}$

Summing up for $k = 1, 2, \dots, n$, and using part (b),

$$\frac{n(2n-1)}{3} + n > \frac{(2n+1)^2}{\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} > \frac{n(2n-1)}{3}$$

$$\Rightarrow \pi^2 \frac{n(2n-1)}{3(2n+1)^2} < \sum_{k=1}^{n} 1/k^2 < \pi^2\left[\frac{n(2n-1)}{3(2n+1)^2} + \frac{n}{(2n+1)^2}\right]$$

By Sandwich, $\lim_{n\to\infty} \sum_{k=1}^{n} 1/k^2 = \frac{\pi^2}{6}$. (Proved)