Theorems on Continuity

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Question: Does there exist a function $f : [0, 1] \to [0, 1)$ which is surjective and continuous?

Solution: We shall show that there does not exist such function. Let us assume to the contrary that f is such a function. Now, for any positive integer n, the number $1 - 1/n$ belongs to $[0, 1)$ which is the range of f, hence there exists $x_n \in [0, 1]$ such that $f(x_n) = 1 - 1/n$. Thus, we get a sequence x_n in $[0, 1]$ such that $f(x_n) = 1 - 1/n$ holds for every $n \geq 1$. This sequence x_n might not converge; but since it is bounded, it has a convergence subsequence (by Bolzano-Weierstrass theorem). Say that subsequence is $\{x_{n_k}\}_{k \geq 1}$ and suppose that it converges to c. Since $0 \leq x_{n_k} \leq 1$ for every $k \geq 1$, so $0 \leq c \leq 1$ (this ensures that f is continuous at c). Now, $x_{n_k} \to c$ and f is continuous, hence $f(x_{n_k}) \to f(c)$. But $f(x_{n_k}) = 1 - 1/n_k$ which converges to 1 as $k \to \infty$. Thus we get $f(c) = 1$, which is a contradiction. □

One key idea in the above solution is to get hold of a sequence x_n such that $f(x_n)$ has a desired property, and then use a convergent subsequence of x_n to derive some contradiction. This idea will be used again and again in this note for proving some theorems on continuous functions.

Theorem. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Then f must be bounded.

Proof. Let, if possible, f be unbounded. Without loss of generality, we may assume that f is unbounded above. Then for every $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $f(x_n) > n$. Now, x_n is bounded, hence it has a convergent subsequence. Say that subsequence is x_{n_k} which converges to c. Since $a \leq x_{n_k} \leq b$ for every $k \geq 1$, it follows that $a \leq c \leq b$. Thus, $x_{n_k} \to c$ and f is continuous at c, which implies that $f(x_{n_k})$ converges to $f(c)$. This in turn implies that $f(x_{n_k})$ must be bounded. But, $f(x_{n_k}) > n_k$ for each $k \geq 1$ and n_k's are strictly increasing, which means that $f(x_{n_k})$ is unbounded[1]. Thus we arrive at a contradiction. □

[1] If a sequence is unbounded above, it need not be true that any subsequence of it must also be unbounded. But here we have $f(x_n) > n$ for every $n \geq 1$. This ensures that the subsequence $f(x_{n_k})$ is also unbounded.

The last theorem says that if a function is continuous on a closed bounded interval, then it must be bounded on that interval. But it does not tell us whether the function attains a maximum or minimum there. However, it turns out that such a function indeed attains its extreme values:

> **Theorem.** Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Then there exists $c, d \in [a, b]$ such that $f(c) \geq f(x) \geq f(d)$ holds for every $x \in [a, b]$.

Proof.(Optional) Let us show that f attains a maximum value (the proof for minimum will be similar). Consider the set $S = \{f(x) : a \leq x \leq b\}$ (which is just the range of f). The last theorem shows that S is bounded above, hence S has a supremum (least upper bound of S). Call $u = \sup S$. Now, for every $n \in \mathbb{N}$, $u - 1/n$ is not an upper bound of S, which means that there is member of S which is greater than $u - 1/n$. Therefore, there exists x_n in $[a, b]$ such that $u - 1/n < f(x_n) \leq u$. Therefore, Sandwich theorem tells us that $f(x_n) \to u$ as $n \to \infty$. Now, x_n might not converge, but it has a convergent subsequence x_{n_k}. Suppose $x_{n_k} \to c$ and it follows that $c \in [a, b]$. Since f is continuous at c, we get $f(x_{n_k}) \to f(c)$. But earlier we had $f(x_n) \to u$. Hence, $f(c) = u = \sup S \implies f(x) \leq f(c)$ for every $x \in [a, b]$. □

The last two theorems can be combined to say the following: if a function is continuous on a closed bounded interval then not only it must be bounded, but it also attains a maximum and a minimum inside that interval. This result is known as the *Extreme Value Theorem*. This theorem will be crucial for some results on differentiation and integration that we shall see in due course of time.

Next, we shall explore another property of continuous function. Suppose a kid measures his height every time he goes to a doctor. On two successive occasions, he measured his height to be 4 ft 2 inches and 4 ft 4 inches respectively. Since height increases continuously w.r.t. time, we can surely say that at some point of time, he was exactly 4 ft 3 inches tall. This is known as the *Intermediate Value Property* (IVP). The next theorem tells us that if a function is continuous on a closed bounded interval then it has this property.

> **Theorem.** Suppose f is a function continuous on $[a, b]$ and $k \in \mathbb{R}$ such that $f(a) < k < f(b)$. Then there exists c between a and b such that $f(c) = k$.

This is known as the Intermediate Value Theorem. In order to prove this theorem, we may assume w.l.o.g. that $k = 0$. Because, once we prove it for $k = 0$, we can apply that to the function $g(x) = f(x) - k$. The proof for $k = 0$ is given below.

Theorem. Suppose f is a function continuous on $[a, b]$ such that $f(a) < 0 < f(b)$. Then there exists c between a and b such that $f(c) = 0$.

Proof.(Optional) Consider $S = \{x : a \le x \le b \text{ and } f(x) \le 0\}$. The set S is bounded above and hence it has a supremum. Let us call $c = \sup S$. We shall show that $f(c) = 0$. Since $a \le c \le b$, it follows that f is continuous at c. Fix any $\varepsilon > 0$. Then there exists $\delta > 0$ such that $f(x) - \varepsilon < f(c) < f(x) + \varepsilon$ holds whenever $x \in (c - \delta, c + \delta)$. Now, the definition of supremum tells us that the interval $(c - \delta, c]$ contains a member of S, call it x_0. And the interval $(c, c + \delta)$ does not contain a member of S, call it y_0. We have $f(c) < f(x_0) + \varepsilon \le \varepsilon$ and $f(c) > f(y_0) - \varepsilon > -\varepsilon$. Therefore, $-\varepsilon < f(c) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $f(c) = 0$. $\square$

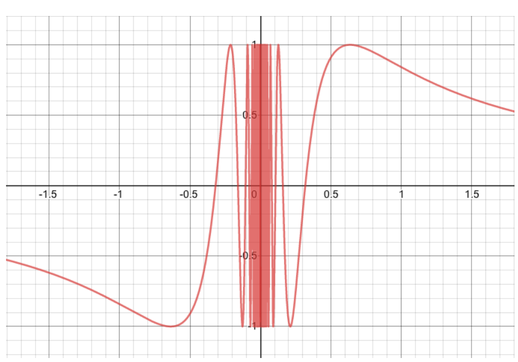
Note, in the above theorem we had $f(a) < 0 < f(b)$. The result holds even if $f(a) > 0 > f(b)$. So we can restate the theorem as: if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then there exists $c \in (a, b)$ such that $f(c) = 0$.

The last theorem is known as Bolzano's theorem. There is an alternate proof of this theorem that essentially does a binary search for finding a root (see section 5.3 in the book 'Introduction to Real Analysis' by Bartle and Sherbert).

An important corollary of the extreme value theorem and intermediate value theorem is the following:

Corollary. A continuous function maps a closed bounded interval into a closed bounded interval. In other words, if $f : [a, b] \to \mathbb{R}$ is continuous, then image of f is $[c, d]$ for some $c < d$.

Let $f : [a, b] \to \mathbb{R}$ be a function. We showed that if f is continuous then f has the intermediate value property (IVP). Now you might ask, is the converse true? It turns out that the answer is 'No'. Here is a counter-example: define $f(x) = \sin(1/x)$ for $x \ne 0$ and $f(0) = 0$.



We can show that f is not continuous at 0. But does f have IVP? Look at the graph above. Take any interval around 0, say$[-\varepsilon, \varepsilon]$. Since f oscillates completely (from -1 to 1) infinitely often, we can show that f attains all the values in $[-1, 1]$. Thus, the above function f has IVP, although it it not continuous at 0.

Let us now solve a problem making use of the intermediate value property.

Problem. Suppose f is a function with $f(0) = f(1)$. If f has IVP, then show that there exists $c \in (0, 1)$ such that $f(c + 1/2) = f(c)$.

Proof. Consider the function $g(x) = f(x + 1/2) - f(x)$, $0 \le x \le 1/2$. Clearly g is continuous. Observe that $g(0) = f(1/2) - f(0)$ and $g(1/2) = f(1) - f(1/2) = f(0) - f(1/2)$. If $g(0)$ or $g(1/2)$ is zero, then the result follows. Else, $g(0)$ and $g(1)$ have opposite signs which implies that there exists $c \in (0, 1/2)$ such that $g(c) = 0 \implies f(c + 1/2) = f(c)$. □

Problem. Let $f : [0, 1] \to [0, 1]$ be continuous. Show that f has a *fixed point* in $[0, 1]$; that is, there exists $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

Proof. Consider $g(x) = f(x) - x$. Since $0 \le f(x) \le 1$ for every $x \in [0, 1]$, we have $g(0) \ge 0$ and $g(1) \le 0$. If $g(0)$ or $g(1)$ equals zero then we are through. Else, we have $g(0) > 0 > g(1)$ and g is continuous on $[0, 1]$, so there exists $x_0 \in (0, 1)$ such that $g(x_0) = 0$. □

Problem. Let $P(x)$ be a polynomial with real coefficients and suppose that the degree of $P(x)$ is odd. Show that the equation $P(x) = 0$ must have at least one real root.

Proof. Suppose $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $(a_n \neq 0)$. We can assume w.l.o.g that $a_n = 1$. Now, it is easy to see that

$$\lim_{x \to \infty} \frac{P(x)}{x^n} = 1 = \lim_{x \to -\infty} \frac{P(x)}{x^n}.$$

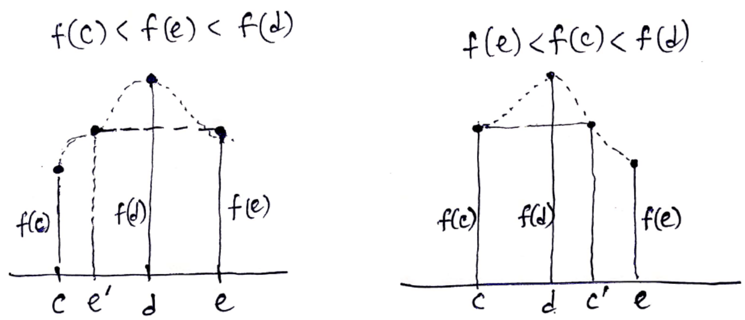
Since n is odd, we can say that $P(x)$ must be positive for some sufficiently large positive x and $P(x)$ must be negative for some sufficiently large negative x. Thus, we get $a < b$ such that $P(a) < 0 < P(b)$. Since $P(x)$ is continuous, there exists c between a, b such that $P(c) = 0$. □

There are plenty of such problems, some are given as exercises at the end of this note. Next we shall see another important result that holds for continuous functions.

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Can we say that f must be monotone? The answer is clearly 'No'. Consider another situation: suppose $f : [a, b] \to \mathbb{R}$ is continuous and one-one. Can we now conclude that f must be monotone? This can be understood better by drawing some pictures, trying to make f continuous and one-one but not monotonic. If you try this yourself, you will get a feeling of why the next theorem holds.

Theorem. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and one-one. Then f must be strictly monotone.

Proof. Fix any $c < d < e$ in the interval $[a, b]$. We shall show that either $f(c) < f(d) < f(e)$ or $f(c) > f(d) > f(e)$ must hold. Let us assume to the contrary that none of them holds. Then we have four other possibilities: (i) $f(c) < f(e) < f(d)$, (ii) $f(e) < f(c) < f(d)$, (iii) $f(d) < f(e) < f(c)$, and (iv) $f(d) < f(c) < f(e)$.



If $f(c) < f(e) < f(d)$ (i.e. (i) holds), then IVP tells us that there exists $e' \in (c, d)$ such that $f(e') = f(e)$. But $e' \neq e$ because $e' < d < e$. This contradicts that f is one-one. In a similar spirit, if $f(e) < f(c) < f(d)$ (i.e. (ii) holds), then there exists $c' \in (d, e)$ such that $f(c') = f(c)$. But $c' \neq c$ because $c < d < c'$. This contradicts that f is one-one. The other two cases are left for the reader.

Having shown the above fact, we are ready to prove that f is monotone. Take any $x < y$ in $[a, b]$. First consider the points $a \leq x < y$. The above fact tells us that either $f(a) \leq f(x) < f(y)$ or $f(a) \geq f(x) > f(y)$ holds. Next, consider the points $x < y \leq b$. We get that either $f(x) < f(y) \leq f(b)$ or $f(x) > f(y) \geq f(b)$ holds. Combining these two, we can say that either $f(a) \leq f(x) < f(y) \leq f(b)$ or $f(a) \geq f(x) > f(y) \geq f(b)$ must hold. Therefore, if $f(a) < f(b)$, then we can say that $x < y \implies f(x) < f(y)$ for every $x, y \in [a, b]$; and if $f(a) > f(b)$, then $x < y \implies f(x) > f(y)$ for every $x, y \in [a, b]$. This completes the proof. $\square$

The last theorem is found to be very useful while solving problems. Two such problems are given below, and many other are included as exercises in the end.

Problem. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfies $|f(x) - f(y)| = 3|x - y|$ for every $x, y \in \mathbb{R}$.

Proof. It is easy to show that f is continuous and one-one. Using this, we can conclude that f must be strictly monotone (see exercise 16 below). This means that either f is strictly increasing, or strictly decreasing. Assume that f is strictly increasing. Then for any x, y we have $f(x) > f(y) \iff x > y$. Hence from the given equation, we deduce that $f(x) - f(y) = 3(x - y) \implies f(x) - 3x = f(y) - 3y$ for every $x, y \in \mathbb{R}$. Set $y = 0$ to get $f(x) = 3x + c$ for every $x \in \mathbb{R}$ where $c = f(0)$. For the case where f is strictly decreasing, we can show that $f(x) = -3x + c$ for all $x \in \mathbb{R}$. These are the only possibilities for f that satisfies the given equation. □

Problem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f^n(x) = (-x)$ for every $x \in \mathbb{R}$. Here f^n denotes f composed with itself n times, e.g. $f^2(x) = f(f(x))$. Prove that n must be odd and find all such functions $f(x)$.

Proof. First note that f is one-one. Since f is continuous, f must be monotone. If f is increasing, then $x > y \implies f(x) > f(y) \implies f^2(x) > f^2(y) \implies \cdots \implies f^n(x) > f^n(y) \implies -x > -y$ which is a contradiction. Therefore f must be decreasing.

Next, if n is even (say $n = 2k$), then we get a similar contradiction: $x > y \implies f(x) < f(y) \implies f^2(x) > f^2(y) \implies f^3(x) < f^3(y) \implies \cdots \implies f^{2k}(x) > f^{2k}(y) \implies -x > -y$ (contradiction). Therefore, n must be odd.

Observe another fact that $f(-x) = f(f^n(x)) = f^n(f(x)) = -f(x)$ for every $x \in \mathbb{R}$. In other words, f is an odd function.

To sum up, we have shown that (i) f is decreasing, (ii) f is odd, (iii) n is odd. We claim that $f(x) = -x$. To prove our claim, fix any $x \in \mathbb{R}$. If $f(x) > -x$, then $f^2(x) < f(-x) = -f(x) < x \implies f^3(x) > f(x) > -x \implies f^4(x) < f(-x) < x$ and so on. Eventually we arrive at $f^n(x) > -x \implies -x > -x$ which is a contradiction. Similar contradiction arises if $f(x) < -x$. Hence we conclude that $f(x) = -x$ for every $x \in \mathbb{R}$. □

Exercises/Problems

1. Let $f : I \to \mathbb{R}$ be a continuous function. If I is a closed bounded interval, then we know that f must be bounded. Show that the result fails to hold in each of the following cases: (a) I is bounded interval but not closed. (b) I is not bounded. (c) I is not an interval.

2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with the property that for every $x \in [a, b]$, there exists $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there exists $c \in [a, b]$ such that $f(c) = 0$.

3. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous and such that $f(a) < g(a)$ and $f(b) > g(b)$. Show that there exists $c \in (a, b)$ such that $f(c) = g(c)$.

4. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Let $x_1, x_2, \ldots, x_n$ be any n points in (a, b). Show that there exists $x_0 \in (a, b)$ such that
$$f(x_0) = \frac{1}{n}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right).$$

5. Prove that the equation $(1 - x)\cos x = \sin x$ has at least one solution in $(0, 1)$.

6. If $f : [0, 1] \to [0, 1]$ is a continuous function, then show that there exists $c \in [0, 1]$ such that $f(c) = c^2$.

7. Suppose that $f : [0, 1] \to [0, 1]$ is a continuous function with $f(0) = 0$ and $f(1) = 1$. Show that there exists $c \in (0, 1)$ such that $c^2 + (f(c))^2 = 1$.

8. Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous and $f(0) = f(2)$. Prove that there exists $a, b \in [0, 2]$ such that $b - a = 1$ and $f(b) = f(a)$.

9. An athlete runs a distance of 6 km in 30 minutes. Prove that somewhere during the run he covered a distance of 1 km in exactly 5 minutes.

10. Consider $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $f(x) = 1/x$. Is f continuous? Note that $f(-1) < 0 < f(1)$ and $f(x) \neq 0$ for any x. Does it contradict the intermediate value theorem?

11. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that there exists x_0 such that $f(x_0 + T/2) = f(x_0)$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that there exists x_0 such that $f(x_0 + \pi) = f(x_0)$. Convince yourself that the same result holds even if we replace π with any other number.

13. Suppose that f and g have the intermediate value property on some closed bounded interval I. Is it necessary that $f + g$ also has the intermediate value property on I?

14. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function that takes rational values only. What can you say about f?

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(q + 1/n) = f(q)$ for every $q \in \mathbb{Q}$ and for every $n \in \mathbb{N}$. Show that f must be a constant function.

16. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and injective. Show that f must be strictly monotonic.

17. Let $f : [a, b] \to [c, d]$ be a strictly increasing function, where $c = f(a), d = f(b)$. Is it necessary that f^{-1} exists?

18. Let $f : [a, b] \to [c, d]$ be a continuous and strictly increasing function, where $c = f(a), d = f(b)$. Show that f^{-1} exists and is strictly increasing on the interval $[c, d]$. Furthermore, show that f^{-1} is continuous on $[c, d]$.

19. Suppose $x_1 = \tan^{-1} 2 > x_2 > x_3 > \cdots$ are positive real numbers, satisfying

$$\sin(x_{n+1} - x_n) + 2^{-(n+1)} \sin x_n \sin x_{n+1} = 0 \text{ for every } n \geq 1.$$

Find an expression for $\cot x_n$. Hence show that $\lim_{n\to\infty} x_n = \frac{\pi}{4}$.

20. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real coefficients, where $n > 0$ is even. If $a_n > 0$ and $a_0 < 0$, then show that the equation $f(x) = 0$ has at least two real roots.

21. Show that there exists a set of 100 consecutive integers of which exactly 19 are primes. (Hint: Do you know that there is a set of 100 consecutive integers which does not contain any prime?)