1. (5+5 points) Let $a_1, a_2, \dots, a_n$ be any n positive real numbers. Calculate the following limits:

$$\text{(i)} \lim_{x\to 0}\left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{1/x}, \quad \text{(ii)} \lim_{x\to\infty}\left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{1/x}.$$

(The answers might involve $a_1, \dots, a_n$, of course!)

Solution. (i) Note that here we have a function $a(x)^{b(x)}$ where $a(x) \to 1$ (as $x \to 0$) and $b(x)$ is unbounded, so it might be useful to take $\log$. With this motivation, we consider:

$$f(x) = \log\left(\left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{1/x}\right) = \frac{1}{x}\log\left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right).$$

First we write

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1}{x}\log\left(1 + \frac{a_1^x - 1 + a_2^x - 1 + \cdots + a_n^x - 1}{n}\right). \tag{1}$$

Now n is fixed, and we have $\lim\limits_{x\to 0}(a_k^x - 1) = 0$ for each $k = 1, \dots, n$. Hence

$$u(x) = \frac{a_1^x - 1 + a_2^x - 1 + \cdots + a_n^x - 1}{n} \to 0, \text{ as } x \to 0,$$

and we know that $\lim\limits_{u\to 0}\dfrac{\log(1+u)}{u} = 1$, so the limit in (1) reduces to

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x}\left(\frac{a_1^x - 1 + a_2^x - 1 + \cdots + a_n^x - 1}{n}\right) &= \frac{1}{n}\sum_{k=1}^{n}\lim_{x\to 0}\frac{a_k^x - 1}{x} && \text{(since } n \text{ is fixed)}\\
&= \frac{1}{n}\sum_{k=1}^{n}\log a_k && \text{(a well-known limit)}\\
&= \log\left((a_1\cdots a_n)^{1/n}\right).
\end{aligned}$$

We have shown that $\lim\limits_{x\to 0} f(x) = \log((a_1\cdots a_n)^{1/n})$. Now it follows by the continuity of the function $t \mapsto e^t$ that the required limit is $(a_1\cdots a_n)^{1/n}$, the geometric mean of $a_1, \dots, a_n$.

(ii) First note that $n^{1/x} \to 1$ when $x \to \infty$, so we can totally ignore the denominator. Let's assume that $\max\{a_1, \dots, a_n\} = a_m$. When we take a_m^x out of the bracket, what remains inside is $1 +$ a small quantity. To make this precise, we will give upper and lower bounds and use Sandwich theorem.

With $a_m = \max\{a_1, \dots, a_n\}$, we have $a_m^x \le \sum_{k=1}^n a_k^x \le n a_m^x$ for any $x > 0$. Therefore,

$$\frac{a_m}{n^{1/x}} \le \left(\frac{\sum_{k=1}^n a_k^x}{n}\right)^{1/x} \le a_m.$$

Since $n^{1/x} \to 1$ as $x \to \infty$, Sandwich theorem tells us that the desired limit is a_m, which is nothing but $\max\{a_1, \dots, a_n\}$.

Remark. You might be aware of the fact that for any $r \neq 0$, we define the r-th power mean of n positive real numbers $a_1, \ldots, a_n$ as

$$M_r = \Big(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\Big)^{1/r}.$$

The above problem gives us the intuition behind why it is customary to define M_0 to be the geometric mean of the n real numbers, and M_∞ to be their maximum. We also define $M_{-\infty}$ to be their minimum. With this general definition, we have the result that $M_r \leq M_s$ for any $-\infty \leq r \leq s \leq \infty$, which is commonly known as the *power mean inequality*.

2. (5 points) Let $P(x)$ be any polynomial with positive real coefficients. Determine, with proof, the following limit:

$$\lim_{x\to\infty} \frac{\lfloor P(x) \rfloor}{P(\lfloor x \rfloor)}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Solution. First observe that if $P(x)$ is a constant polynomial, say c, the limit will obviously be $\lfloor c \rfloor / c$. Let us now assume that $P(x)$ has degree $n > 1$.

Note that $P(x)$ is an increasing function of x. To see why, suppose that $P(t) = \sum_{k=0}^n a_k t^k$, and consider any $x < y$. Note that $P(y) - P(x) = \sum_{k=0}^n a_j (y^k - x^k)$, which is positive because each summand is positive. Moreover, we can see that $P(x) > 0$ for every $x > 0$. Next, we use the bounds $t - 1 < \lfloor t \rfloor \leq t$ to obtain

$$\frac{P(x)-1}{P(x)} \leq \frac{P(x)-1}{P(\lfloor x \rfloor)} < \frac{\lfloor P(x) \rfloor}{P(\lfloor x \rfloor)} \leq \frac{P(x)}{P(\lfloor x \rfloor)} < \frac{P(x)}{P(x-1)}. \tag{2}$$

Now if we let $x \to \infty$, on one side we have

$$\lim_{x\to\infty} \frac{P(x)-1}{P(x)} = \lim_{x\to\infty} \Big(1 - \frac{1}{\sum_{k=0}^n a_j x^n}\Big) = 1.$$

And on the other side,

$$\begin{aligned}
\lim_{x\to\infty} \frac{P(x-1)}{P(x)} &= \lim_{x\to\infty} \Big(1 - \frac{P(x) - P(x-1)}{P(x)}\Big) \\
&= 1 - \lim_{x\to\infty} \frac{\text{poly. in } x \text{ of degree} \leq n-1}{\text{poly. in } x \text{ of degree } n} \\
&= 1 - \lim_{x\to\infty} \frac{x^{n-1}\,(b_{n-1} + b_{n-2}/x + b_{n-3}/x^2 + \cdots + b_0)}{x^n\,(a_n + a_{n-1}/x + a_{n-2}/x^2 + \cdots + a_0)} \\
&= 1.
\end{aligned}$$

Therefore both the LHS and RHS of (2) go to 1 as $x \to \infty$, and hence we can apply Sandwich theorem to conclude that the desired limit (exists and) equals 1.

3. (10 points) Suppose that $f : [1, 2] \to \mathbb{R}$ is a continuous function that satisfies

$$f(x) = \sum_{n=1}^{\infty} \frac{f(x^{1/n})}{2^n}$$

for every $x \in [1, 2]$. Show that f must be a constant function.

Solution. Since f is a continuous function on the closed bounded interval $[1, 2]$, we know by the *Extreme Value Theorem* that f attains a minimum and maximum value, say at $x = a$ and $x = b$, respectively. This implies that

$$f(a) \leq f(x) \leq f(b) \text{ holds for every } x \in [1, 2]. \tag{3}$$

Putting $x = a$ in the given property of f we get

$$\sum_{n=1}^{\infty} \frac{f(a^{1/n})}{2^n} = f(a) = \sum_{n=1}^{\infty} \frac{f(a)}{2^n}. \tag{4}$$

Note that (3) tells us that $f(a^{1/n}) \geq f(a)$ for each $n \geq 1$. So the above equality would be possible only if

$$f(a^{1/n}) = f(a) \text{ for every } n \geq 1.$$

(Otherwise the LHS of (4) would be strictly less than the RHS.) Now if we let $n \to \infty$, and use the continuity of f, we get

$$f(a) = \lim_{n\to\infty} f(a^{1/n}) = f(\lim_{n\to\infty} a^{1/n}) = f(1).$$

A similar argument holds for $f(b)$ as well (since we have $f(b^{1/n}) \leq f(b)$ for each $n \geq 1$), which gives us $f(b) = f(1)$. Thus, the minimum and the maximum values of f on the interval $[1, 2]$ have to be equal, which makes f a constant function. It is easy to check that any constant function on $[1, 2]$ satisfies the given property.

4. (10 points) Find all values of $\theta > 0$ for which the following series converges:

$$\sum_{n=1}^{\infty} \left(\sqrt[\theta]{n^\theta + 1} - n \cos \frac{1}{n^{\theta/2}} \right)^\theta.$$

Solution. The key idea here is to write

$$\left(\sqrt[\theta]{n^\theta + 1} - n \cos \frac{1}{n^{\theta/2}} \right)^\theta = \left(\sqrt[\theta]{n^\theta + 1} - n + n - n \cos \frac{1}{n^{\theta/2}} \right)^\theta. \tag{5}$$

Observe that

$$\sqrt[\theta]{n^\theta + 1} - n = n \left((1 + 1/n^\theta)^{1/\theta} - 1 \right) = n^{1-\theta} \cdot \frac{(1 + 1/n^\theta)^{1/\theta} - 1}{1/n^\theta}.$$

Now $\theta > 0$, so $1/n^\theta \to 0$ as $n \to \infty$, and we know that $\lim_{x\to 1} \frac{x^\nu - 1^\nu}{x-1} = \nu$. Hence

$$\lim_{n\to\infty} \frac{\sqrt[\theta]{n^\theta + 1} - n}{n^{1-\theta}} = \frac{1}{\theta}. \tag{6}$$

Next, for the second part, we note that

$$\lim_{n\to\infty} \frac{n - n\cos(n^{-\theta/2})}{n^{1-\theta}} = \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}. \tag{7}$$

Informally, (6) tells us that $(\sqrt[\theta]{n^\theta+1} - n) \approx \frac{1}{\theta} \cdot n^{1-\theta}$ and $(n - n\cos(n^{-\theta/2})) \approx \frac{1}{2} n^{1-\theta}$. Therefore, the given sum should behave similar to $\sum_{n\geq 1} n^{\theta(1-\theta)}$.

To make this idea rigorous, let us fix $\varepsilon > 0$, smaller than $\theta/2$ and $1/2$. Invoking (6) and (7) we can say that for all sufficiently large n, say for every $n \geq n_0$, the following holds

$$\left| \frac{\sqrt[\theta]{n^\theta + 1} - n}{n^{1-\theta}} - \frac{1}{\theta} \right| < \varepsilon, \text{ and } \left| \frac{n - n\cos(n^{-\theta/2})}{n^{1-\theta}} - \frac{1}{2} \right| < \varepsilon.$$

Using triangle inequality, we easily deduce from here that

$$\left(\frac{1}{\theta} + \frac{1}{2} - 2\varepsilon\right) n^{1-\theta} < \sqrt[\theta]{n^\theta + 1} - n\cos(n^{-\theta/2}) < \left(\frac{1}{\theta} + \frac{1}{2} + 2\varepsilon\right) n^{1-\theta}$$

for every $n \geq n_0$. The above bounds tell us that the given series converges if and only if the series $\sum_{n\geq 1} n^{\theta(1-\theta)}$ converges.

This is due to the following result: If $0 < a_n < b_n$ holds for all sufficiently large n, then

(a) $\sum_{n=1}^\infty b_n$ convergent implies that $\sum_{n=1}^\infty a_n$ must also converge.

(b) $\sum_{n=1}^\infty a_n$ divergent implies that $\sum_{n=1}^\infty b_n$ must also diverge.

Now it is well-known that the series $\sum_{n\geq 1} n^{-\beta}$ converges if and only if $\beta > 1$. Hence $\sum_{n\geq 1} n^{\theta(1-\theta)}$ converges if and only if $\theta(\theta - 1) > 1$. Simple manipulation shows that this is equivalent to $\theta > (1 + \sqrt{5})/2$ (since $\theta > 0$ is given). Therefore we conclude that the given series converges if and only if $\theta > (1+\sqrt{5})/2$.

Remark. The ideas used in this problem are very useful for determining the convergence of a series. For a given series $\sum a_n$, where $a_n > 0$, we wish to compare it with another series $\sum b_n$ where b_n is supposed to be much more simpler to handle than a_n. This comparison can be made rigorous once you can say that $c < a_n/b_n < C$ holds for all sufficiently large n, for some positive constants c and C. If you only have $a_n/b_n < C$ then you can only comment that $\sum a_n$ converges when $\sum b_n$ converges, but not the other way around. If you only have $c < a_n/b_n$ then you can only say that $\sum a_n$ diverges when $\sum b_n$ diverges, but not the other way around. These three notions are often denoted as $a_n = \Theta(b_n), a_n = O(b_n)$, and $a_n = \Omega(b_n)$ respectively. You will learn about them later in greater detail.