Ramanujan School of Mathematics

Class Test 2 on Calculus

Time allotted: 1.5 hours Total points: 35

Attempt all the questions. You can use any result discussed in the class, but you have to state it properly. Since it is a 'take-home' exam, I can only request you to give the test honestly and abide by the time limit. Do not cheat to yourself. All the best!

1. (5+5 points) Let $a_1, a_2, \ldots, a_n$ be any n positive real numbers. Calculate the following limits:

   (i) $\lim_{x\to 0}\left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{1/x}$, (ii) $\lim_{x\to\infty}\left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{1/x}$.

   (The answers might involve $a_1, \ldots, a_n$, of course!)

2. (5 points) Let $P(x)$ be any polynomial with positive real coefficients. Determine, with proof, the following limit:

$$\lim_{x\to\infty} \frac{\lfloor P(x)\rfloor}{P(\lfloor x\rfloor)}$$

   where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.

3. (10 points) Suppose that $f : [1,2] \to \mathbb{R}$ is a continuous function that satisfies

$$f(x) = \sum_{n=1}^{\infty} \frac{f(x^{1/n})}{2^n}$$

   for every $x \in [1,2]$. Show that f must be a constant function.

4. (10 points) Find all values of $\theta > 0$ for which the following series converges:

$$\sum_{n=1}^{\infty} \left(\sqrt[\theta]{n^\theta + 1} - n\cos\frac{1}{n^{\theta/2}}\right)^\theta.$$

   (You may use this result: $\sum_{n=1}^{\infty} n^{-\beta}$ converges iff $\beta > 1$. You need not prove it here.)