Some Computational Problems on Limits and Continuity

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1. Suppose that $\lim_{x\to a}(f(x)+g(x)) = 2$ and $\lim_{x\to a}(f(x)-g(x)) = 1$. Is it necessary that $\lim_{x\to a} f(x)g(x)$ exists? If yes, can you calculate that limit?

2. Find the following limits: (i) $\lim_{x\to 1} \frac{\sqrt{1-\cos 2(x-1)}}{x-1}$, (ii) $\lim_{x\to 1} \frac{x^3 - x^2\log x + \log x - 1}{x^2-1}$, (iii) $\lim_{x\to 1} \frac{\sqrt[3]{x}+\sqrt{x}+x\sqrt{x}-3}{x-1}$

3. Define a function $f(x) = \begin{cases} 5x-4, & \text{if } 0 < x \le 1, \\ 4x^3 - 3x, & \text{if } 1 < x < 2. \end{cases}$ Find out whether $\lim_{x\to 1} f(x)$ exists. Is f continuous at $x = 1$?

4. Let $f(x) = \frac{3x^2+ax+a+1}{x^2+x-2}$. Determine the value(s) of a for which (i) $\lim_{x\to 1} f(x)$ exist, (ii) $\lim_{x\to -2} f(x)$ exists. Do you have any conclusion? Also calculate those two limits.

5. Determine $\lim_{x\to \pi/2} \frac{\lfloor x/2 \rfloor}{\log(\sin x)}$, where $\lfloor\cdot\rfloor$ is the box/floor function.

6. Determine $\lim_{x\to 1} \frac{x^{p+1}-(p+1)x+p}{(x-1)^2}$ where p is a positive integer.

7. Evaluate: (i) $\lim_{x\to y} \frac{\sin^2 x - \sin^2 y}{x^2-y^2}$, (ii) $\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2}$, (iii) $\lim_{x\to \pi/4} \frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x}$.

8. Calculate the following limits: $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$ and $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)$. Also find $\lim_{n\to\infty} n\sin(2\pi\sqrt{1+n^2})$.

9. Calculate the following limits:

   (a) $\lim_{x\to a} \frac{x\sin a - a\sin x}{x-a}$

   (b) $\lim_{\theta\to \pi/4} \frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

   (c) $\lim_{x\to \pi/2} \frac{\sin(\cot^2 x)}{(\pi-2x)^2}$

   (d) $\lim_{x\to\infty} \cot^{-1}(\sqrt{x+1}-\sqrt{x})$.

   (e) $\lim_{x\to \pi/4} \frac{(\cos x+\sin x)^3 - 2\sqrt{2}}{1-\sin 2x}$

   (f) $\lim_{x\to 0} \frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$

10. Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ for the following function:

$$f(x) = \begin{cases} \tan^2(\{x\})/(x^2-[x]^2), & \text{if } x > 0, \\ 1, & \text{for } x = 0, \\ 1/\sqrt{\{x\}\cot\{x\}}, & \text{for } x < 0. \end{cases}$$

Here $[x]$ is the floor function and $\{x\} = x - [x]$.

11. If α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, prove that

$$\lim_{x\to\beta} \frac{1-\cos(ax^2+bx+c)}{(x-\beta)^2} = \frac{1}{2}(b^2-4ac).$$

12. Find whether the limit $\lim_{x\to 0} \frac{\tan(\{x\}-1)\sin(\{x\})}{\{x\}(\{x\}-1)}$ exists.

13. Discuss the continuity of the function $f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$ at $x = 1$.

The next few problems will involve the concept of $e := \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. This number e is a real number, just like any other real number, say $2, -1, 2/3, \sqrt{3}$, or $\pi/2$. Then you might wonder, why is e so special? Actually it is the function $\exp(x) := e^x$ that makes e so special. Here is a note that attempts to give some insights about different results related to e that we usually use in a first course of Calculus. In particular, we assume the following limit without proof:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

A proof of this is given in the above note, you may read that once you are ready. (The way I treated it in the note requires the knowledge of Fundamental theorem of Integral Calculus). Assuming the above limit (and the above definition of e), you can prove the following limits:

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1, \ \lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, \ \lim_{x\to 0}(1+x)^{1/x} = e.$$

Result. If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a}(1+f(x))^{g(x)} = \exp\left(\lim_{x\to a} f(x)g(x)\right)$. (Prove it yourself!) This result is often used when we know that $\lim_{x\to a} a(x) = 1$, and we wish to find $\lim_{x\to a} a(x)^{b(x)}$. We just do this: $\lim_{x\to a} a(x)^{b(x)} = \exp\left(\lim_{x\to a} b(x)\log a(x)\right) = \exp\left(\lim_{x\to a}(a(x)-1)b(x)\right)$.

Key idea: Whenever you have $a(x)^{b(x)}$, take log.

14. $\lim_{x\to 0} \frac{p^x - q^x}{r^x - s^x}$

15. $\lim_{h\to 0} \frac{\log(1+2h) - 2\log(1+h)}{h^2}$

16. $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}$

17. $\lim_{x\to\infty} \left(\frac{x+6}{x+1}\right)^{x+4}$

18. $\lim_{x\to 0} \left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x}$

19. $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$

20. $\lim_{x\to\infty} \left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x$

21. $\lim_{x\to 0} (\cos x)^{\cot^2 x}$