Limit of a function

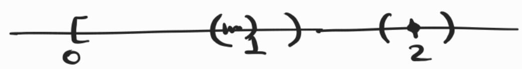
Def^n (limit point)

it not necessary that $a \in D$

A point a is said to be a limit point of $D \subseteq \mathbb{R}$ if every ε-nbd of a contains some element of D other than a.

Example: $[0,1) \cup \{2\} \cup (3,4]$ has limit points: $[0,1] \cup [3,4]$.

$\mathbb{Q} \xrightarrow[\text{pts}]{\text{limits}} \mathbb{R}$



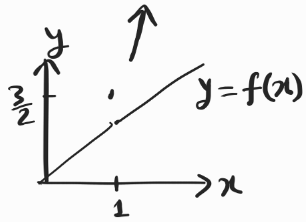
$\{1+\frac{1}{n} : n \geq 1\}$

only limit pt: 1

Def^n (limit of a function)

Suppose $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$. For a limit point a of D, we define $\lim_{x \to a} f(x)$ as follows. We say that this limit exists and equals L if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta,\ x \in D \implies |f(x) - L| < \varepsilon.$$



$$f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 3/2 & \text{if } x = 1 \end{cases} \qquad \lim_{x \to 1} f(x) = 1.$$

Def^n (sequential def^n)

$f : D \subseteq \mathbb{R}$, $D \subseteq \mathbb{R}$. We say that $\lim_{x \to a} f(x) = L$ if for every seq. $\{x_n\} \subseteq D$ s.t. $x_n \to a$ as $n \to \infty$ and $x_n \neq a$ for all $n \geq 1$, it holds that $f(x_n) \to L$ as $n \to \infty$.

Result The above two definitions are equivalent.

ε-δ def^n $\implies$ sequential def^n

Fix $\varepsilon > 0$. $\exists\, \delta > 0$ s.t. .... Let x_n be a seq (in D) s.t. $x_n \to a$, $x_n \neq a$.

$0 < |x_n - a| < \delta \implies |f(x_n) - L| < \varepsilon$.

Sequential def^n $\implies$ ε-δ def^n

Proceed by contradiction. $\exists\, \varepsilon_0 : \forall\, \delta > 0\ \exists\, x : 0 < |x-a| < \delta$ and $|f(x) - L| > \varepsilon_0$.

Since $\delta > 0$ is arbitrary, get hold of a seq. and get a contradiction.

Properties

① Limit of a function, if exists, must be unique.

If L_1, L_2 both are values of a limit, say $\lim_{x\to a} f(x)$, say w.l.o.g. that $L_1 < L_2$, then pick $\varepsilon = L_2 - L_1$ and for x sufficiently close to a, we get

$$|L_2 - L_1| < |f(x) - L_1| + |f(x) - L_2| < \varepsilon/2 + \varepsilon/2 = L_2 - L_1,$$

which is a contradiction.

② If $\lim_{x\to a} f(x) = L$ then $\exists\, \delta > 0$ such that f is bdd within $(a-\delta, a+\delta)$.

Fix some $\varepsilon > 0$, say $\varepsilon = 1$.

$\exists\, \delta > 0$ s.t. $0 < |x-a| < \delta$ (and $x \in D$) $\Rightarrow |f(x) - L| < 1$

$$\Rightarrow |f(x)| \le |L| + |f(x) - L| < |L| + 1.$$

③ If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exists, say L_1 and L_2,

then
$$\lim_{x\to a} (f(x) \pm g(x)) = L_1 \pm L_2,$$
$$\lim_{x\to a} f(x)\, g(x) = L_1 L_2.$$

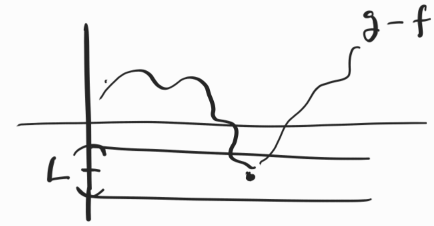
Furthermore, if $g(x) \neq 0$ in a nbd of a and also $L_2 \neq 0$, then $\lim_{x\to a} f(x)/g(x) = L_1/L_2$.

Proof Do yourself. Here is an example/idea of a proof:

"For sufficiently close to a",

$$|f(x)\, g(x) - L_1 L_2| = |f(x)(g(x) - L_2) + L_2 (f(x) - L_1)|$$
$$\le \underbrace{|f(x)|}_{\boxed{< M}}\, \underbrace{|g(x) - L_2|}_{< \varepsilon/2M} + L_2 \underbrace{|f(x) - L_1|}_{< \varepsilon/2L_2} < \varepsilon.$$

④ If $\lim_{x\to a} f(x) = L_1$, $\lim_{x\to a} g(x) = L_2$ and if $f(x) \leq g(x)$ holds in a nbd of a, then $L_1 \leq L_2$.



Proof Let's work with $g(x) - f(x)$
$= h(x)$ (say).

$$\lim_{x\to a} h(x) = L_2 - L_1 = L \text{ (say)}$$

Let, if possible, $L < 0$. Pick $\varepsilon > 0$ s.t. $L + \varepsilon < 0$.

$\{(L-\varepsilon, L+\varepsilon) \subseteq$ negative y axis

$\exists\, \delta > 0$ s.t.

$$0 < |x - a| < \delta \Rightarrow |h(x) - L| < \varepsilon$$

$$\Rightarrow L - \varepsilon < h(x) < L + \varepsilon < 0.$$

But $h(x) \geq 0$, so we get a contradiction.

⑤ (Sandwich theorem)

If f, g, h are functions on $D \subseteq \mathbb{R}$, such that

$$f(x) \leq g(x) \leq h(x)$$

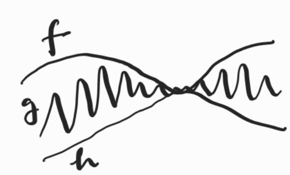
(at least in a nbd of a, where a is a limit pt of D)

and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then $\lim_{x\to a} g(x)$ exists and also equals L.

Proof Fix any $\varepsilon > 0$. $\exists\, \delta_1 > 0$ such that



$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon,$$

and $\exists\, \delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \Rightarrow |h(x) - L| < \varepsilon.$$

Then for $\delta = \min(\delta_1, \delta_2) > 0$, we can say that

$$0 < |x - a| < \delta \Rightarrow \underline{L - \varepsilon} \leq f(x) \leq \underline{g(x)} \leq h(x) \leq \underline{L + \varepsilon}$$

$$\Rightarrow |g(x) - L| < \varepsilon.$$

Since this holds for every $\varepsilon > 0$, $\lim_{x\to a} g(x) = L$. $\blacksquare$

Example

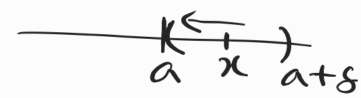
$$\lim_{x \to 0} x \sin \frac{1}{x}.$$

$$-|x| < x \sin \frac{1}{x} < |x|$$

Apply Sandwich.

One sided limits

We say that $\lim_{x \to a^+} f(x) = L_1$ if

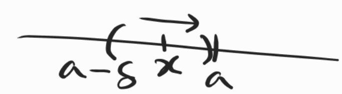


for every $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$0 < x - a < \delta \Rightarrow |f(x) - L_1| < \varepsilon.$$

(i.e., $x \in (a, a+\delta) \cap D$)

We say that $\lim_{x \to a^-} f(x) = L_2$ if



for every $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$0 < a - x < \delta \Rightarrow |f(x) - L_2| < \varepsilon.$$

(i.e., $x \in (a-\delta, a) \cap D$)

Theorem

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

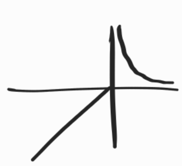
Proof. Discussed verbally. You should be able to finish it yourself.

When limit does not exist at $x = a$

If at least one of the one sided limits does ~~is~~ not exist, or if the two one sided limits exist but are unequal.

e.g.,

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1, \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1, \text{ so } \lim_{x \to 0} \frac{|x|}{x} \text{ does not exist.}$$



$$f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ x & \text{if } x \le 0 \end{cases}$$

$\lim_{x \to 0^-} f(x) = 0$, $\lim_{x \to 0^+} f(x)$ does not exists finitely.

Here $\lim_{x \to 0} f(x)$ does not exist.

Result f is continuous at a ($\in$ Domain) iff

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a).$$

Types of discontuity

(i) When $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \neq f(a)$ (Removable discont.)

This discont. can be removed by redefining $f(a) := L$.

(ii) When the two one sided limits exist, but are unequal. (Jump discont.)

(iii) When at least one of the one sided limits does not exist.

Examples (i) $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(ii) $f(x) = \dfrac{|x|}{x}$ if $x \neq 0$, $f(0) = 0$.

$f(x) = \lfloor x \rfloor$.

(iii) $f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ x & \text{if } x \leq 0 \end{cases}$, $\sin\left(\frac{1}{x}\right)$ etc.

Result

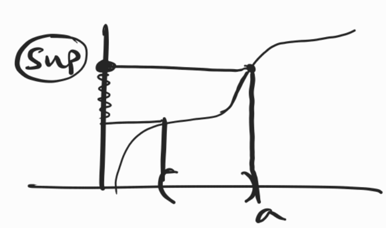
✓ Monotone functions can have only jump discontinuities.

✓ Main idea: One sided limits must exist for monotone functions.

Proof Let's show that $\lim_{x\to a^-} f(x)$ exists. Other side will be similar.

Assume w.l.o.g. that f is inc.

Consider $\{f(x) : x < a\}$. Note that it is bdd above (even if $f(a)$ itself is undefined, but f is defined on some $(a-\delta, a+\delta)\setminus\{a\}$).



$L = \sup\{f(x) : x < a\}$

$\forall \varepsilon > 0$, $L - \varepsilon$ is not an upper bound of this set,
implying that for some $\delta > 0$, $f(a-\delta) > L - \varepsilon$.
Now since f is inc, for every $x \in (a-\delta, a) \cap D$,
we have

$$L - \varepsilon < f(a-\delta) \le f(x) \le L$$

$\therefore$ $\forall \varepsilon > 0$, we got $\delta > 0$ s.t. $0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Example $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = ?$ Call $f(x) = \sin\frac{1}{x}$, $x \neq 0$.

Take $x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, $n \ge 1$.

Then $x_n \to 0$, $y_n \to 0$ and $x_n \neq 0$, $y_n \neq 0$ for all $n \ge 1$,

and $\lim_{n\to\infty} f(x_n) = 0$, whereas $\lim_{n\to\infty} f(y_n) = 1$.

Therefore, $\lim_{x\to 0} f(x)$ does not exist.

Some common limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \lim_{x\to 0}\frac{e^x - 1}{x} = 1,$$

$$\lim_{x\to 0}\frac{\tan x}{x} = 1, \quad \lim_{x\to 0}\frac{\log(1+x)}{x} = 1,$$

$$\lim_{x\to 0}\frac{a^x - 1}{x} = \log a, \text{ etc.}$$

$$e := \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

You need to know (actually memorize) such formulae.
But rest assured that all of them can be proved.
e.g. prove $\lim_{x\to 0}\frac{\sin x}{x}$ by Sandwich.