(3.) $f : [1,2] \to \mathbb{R}$ continuous.

By EVT, f attains a maximum value and a minimum value within $[1,2]$.

Suppose that for $a, b \in [1,2]$,

$$f(a) \le f(x) \le f(b) \longrightarrow (\star)$$

holds for every $x \in [1,2]$.

$$f(a) = \sum_{n=1}^{\infty} \frac{f(a^{1/n})}{2^n} \quad \text{(given)}$$

$\circledast$ $f(a^{1/n}) \ge f(a)$ for each $n \ge 1$ (by $(\star)$)

$$f(a) = \sum_{n=1}^{\infty} \frac{f(a^{1/n})}{2^n} \ge \sum_{n=1}^{\infty} f(a)/2^n = f(a)$$

should be equality

But $\circledast$ holds. So this equality is possible iff $f(a^{1/n}) = f(a)$ for each $n \ge 1$.

Then $f(a) = \lim_{n\to\infty} f(a^{1/n}) = f\left(\lim_{n\to\infty} a^{1/n}\right)$

(∵ f is cont.) on $[1,2]$

$= f(1)$.

By similar argument, show that

$$f(b) = f(1).$$

Therefore, $f(a)$ (min) and $f(b)$ (max) are equal $\Rightarrow$ f is a constant func.

$x_n = a^{1/n} \to 1$ as $n \to \infty$.

f is cont. at 1.

$\therefore f(x_n) \to f(1)$.

$$\lim_{n\to\infty} f(a^{1/n}) = f\left(\lim_{n\to\infty} a^{1/n}\right).$$

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(4) $\sqrt[\theta]{n^\theta + 1} - n\cos\left(n^{-\theta/2}\right)$

$$= \underline{\left(\sqrt[\theta]{n^\theta+1} - n\right)} + \underline{n\left(1 - \cos\left(n^{-\theta/2}\right)\right)}$$

When x is small,

$$1 - \cos x \approx x^2/2$$

$$\left[1 - \cos x = 2\sin^2\frac{x}{2} \approx 2\left(\frac{x}{2}\right)^2 = \frac{1}{2}x^2\right]$$

Precisely: $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

$\therefore$ When n is large,

$$n\left(1-\cos\left(n^{-\theta/2}\right)\right) \approx n \cdot \frac{1}{2}\left(n^{-\theta/2}\right)^2$$
$$= \frac{1}{2} n^{1-\theta}.$$

Precisely,

$$\lim_{n\to\infty} \frac{n\left(1-\cos\left(n^{-\theta/2}\right)\right)}{n^{1-\theta}} = \frac{1}{2}. \quad (*)$$

On the other side,

$$\sqrt[\theta]{n^\theta+1} - n = n\left(\left(1+n^{-\theta}\right)^{1/\theta} - 1\right)$$

$$= n^{1-\theta} \times \frac{\left(1+n^{-\theta}\right)^{1/\theta} - 1}{n^{-\theta}}$$

$$\lim_{n\to\infty} \frac{\left(1+n^{-\theta}\right)^{1/\theta}-1}{n^{-\theta}} = \lim_{x\to 0} \frac{(1+x)^{1/\theta}-1}{x}$$

$$= 1/\theta.$$

Therefore,

$$\lim_{n\to\infty} \frac{\sqrt[\theta]{n^\theta+1}-n}{n^{1-\theta}} = \frac{1}{\theta}. \quad (**)$$

Def^n of $f'(a)$:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

$$\lim_{h\to 0} \frac{(1+h)^{1/\theta}-1}{h} = \text{derivative of } x^{1/\theta} \text{ at } x=1.$$

$$= \text{derivative of } e^{1/\theta \log x} \text{ at } x=1$$

$$= e^{1/\theta \log x} \frac{1}{x\theta} \Big|_{x=1}$$

$$= \frac{1}{\theta}.$$

$$\lim_{x\to 0} \frac{(1+x)^{1/\theta}-1}{x}$$

$$\lim_{z\to 0} \frac{e^z-1}{z} = 1$$

$$= \lim_{x\to 0} \frac{e^{1/\theta \log(x+1)}-1}{\frac{1}{\theta}\log(x+1)} \times \frac{1}{\theta} \frac{\log(x+1)}{x}$$

$$= \lim_{z\to 0} \frac{e^z-1}{z} \times \frac{1}{\theta} \times \lim_{x\to 0} \frac{\log(x+1)}{x}$$

$$= \frac{1}{\theta}.$$

$\left[z = \frac{1}{\theta}\log(x+1) \to 0 \text{ as } x \to 0\right]$

Note that (*) and (**) tell us that for all sufficiently large n,

$$n\left(1-\cos\left(n^{-\theta/2}\right)\right) \approx \frac{1}{2} n^{1-\theta}$$

and

$$\sqrt[\theta]{n^\theta+1} - n \approx \frac{1}{\theta} n^{1-\theta}.$$

Fix $\varepsilon > 0$, small s.t. $1/\theta > \varepsilon$, $1/2 > \varepsilon$.

Then (*) and (**) say that for all sufficiently large n,

$$\left| \frac{n\left(1-\cos\left(n^{-\theta/2}\right)\right)}{n^{1-\theta}} - \frac{1}{2} \right| < \varepsilon,$$

and

$$\left| \frac{\sqrt[\theta]{n^\theta+1} - n}{n^{1-\theta}} - \frac{1}{\theta} \right| < \varepsilon.$$

How to combine these two?

$$\left| \frac{\sqrt[\theta]{n^\theta+1} - n}{n^{1-\theta}} + \frac{n(1-\cos(n^{-\theta/2}))}{n^{1-\theta}} - \left(\frac{1}{\theta}+\frac{1}{2}\right) \right|$$

$$\leq \left| \frac{\sqrt[\theta]{n^\theta+1} - n}{n^{1-\theta}} - \frac{1}{\theta} \right| + \left| \frac{n(1-\cos(n^{-\theta/2}))}{n^{1-\theta}} - \frac{1}{2} \right|$$

$$< 2\varepsilon.$$

$$a_n = \sqrt[\theta]{n^\theta+1} - \cos(n^{-\theta/2}).$$

$$\left| \frac{a_n}{n^{1-\theta}} - \left(\frac{1}{\theta}+\frac{1}{2}\right) \right| < 2\varepsilon$$

$$\Rightarrow -2\varepsilon < \frac{a_n}{n^{1-\theta}} - \left(\frac{1}{\theta}+\frac{1}{2}\right) < 2\varepsilon$$

$$\Rightarrow \left(\frac{1}{2}+\frac{1}{\theta}-2\varepsilon\right)n^{1-\theta} < a_n < \left(\frac{1}{2}+\frac{1}{\theta}+2\varepsilon\right)n^{1-\theta}$$

for <u>all</u> sufficiently large n.

$\therefore \sum_{n\geq 1} a_n^\theta$ converges iff $\sum_{n\geq 1} n^{\theta(1-\theta)}$ converges.

$c_n < a_n < b_n$ for all $n \geq n_0$ $\quad \sum b_n$ converges $\Rightarrow \sum a_n$ converges

$\sum c_n$ diverges $\Rightarrow \sum a_n$ diverges

$\therefore$ The given series converges iff

$$\sum_{n=1}^{\infty} n^{-\theta(\theta-1)} \text{ converges.}$$

$$\Updownarrow$$

$$\theta(\theta-1) > 1$$

$$\Updownarrow$$

$$\theta > \frac{1+\sqrt{5}}{2} \quad (\because \theta > 0 \text{ is given})$$
