Fermat's theorem and the Mean Value Theorems

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1. Suppose that $f : [a, b] \to \mathbb{R}$ has a local maximum at $x = c$, where $c \in (a, b)$. Is it necessary that $f'(c) = 0$?

2. Suppose that $f : [a, b] \to \mathbb{R}$ has a local maximum at $x = c$, where $c \in [a, b]$. Assume that f is differentiable at $x = c$. Is it necessary that $f'(c) = 0$?

3. (*Fermat's Theorem*) Suppose that $f : [a, b] \to \mathbb{R}$ has a local maximum (or minimum) at $x = c$, where $c \in (a, b)$. Assume that f is differentiable at $x = c$. Show that $f'(c)$ must be equal to zero.

   (Note: It is enough to show it only for the case of local maximum. If f has a local minimum at $x = c$, then we can just apply that result to $-f$.)

4. Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable at $x = c$, where $c \in (a, b)$. If $f'(c) = 0$, is it necessary that f has a local maximum/minimum at $x = c$?

5. (*Rolle's Theorem*) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Prove that there exists $c \in (a, b)$ such that $f'(c) = 0$.

6. Show that Rolle's theorem fails to hold if f is not continuous at one of endpoints (a or b).

7. (*Lagrange's Mean Value Theorem*) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

8. Let f be continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$. Prove that for any $\beta \in \mathbb{R}$, there exists some $c \in (a, b)$ such that $f'(c) + \beta \cdot f(c) = 0$.

9. Let f, g be continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$. Show that there exists $c \in (a, b)$ such that $g'(x)f(x) + f'(x) = 0$.

10. Let f be continuous on $[0, \pi]$, differentiable on $(0, \pi)$. Show that there exists $c \in (0, \pi)$ such that $f'(c) \sin c + f(c) \cos c = 0$.

11. Assume that f is continuous on $[a, b]$, differentiable on (a, b), where $0 < a < b$, and suppose that $bf(a) = af(b)$. Prove that there exists $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

12. Suppose that f is continuous on $[a, b]$, differentiable on (a, b), and suppose that $f(b)^2 - f(a)^2 = b^2 - a^2$. Prove that the equation $f'(x)f(x) = x$ has at least one root in (a, b).

13. Let f and g be continuous and never vanishing on $[a, b]$ and differentiable on (a, b). Prove that if $f(a)g(b) = f(b)g(a)$ then there is $x_0 \in (a, b)$ such that
$$\frac{f'(x_0)}{f(x_0)} = \frac{g'(x_0)}{g(x_0)}.$$

14. Assume that $a_0, a_1, \ldots, a_n$ are real numbers such that
$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$
Show that $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ has at least one root in $(0, 1)$.

15. Let $P(x)$ be a polynomial with real coefficients and degree $n \geq 2$. Prove that if all roots of $P(x)$ are real, then all roots of $P'(x)$ are also real.

16. Let f be continuous on $[0, 2]$ and twice differentiable on $(0, 2)$. If $f(x) = x$ holds for $x = 0, 1$ and 2, then show that there exists $x_0 \in (0, 2)$ such that $f''(x_0) = 0$.

17. Let $n > 1$ be an integer, and let $f : [a, b] \to \mathbb{R}$ be a continuous function, which is n-times differentiable on (a, b). If the graph of $f(x)$ has $n + 1$ collinear points, then prove that there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$.
[Here $f^{(n)}(c)$ denotes the n-th derivative of $f(x)$, evaluated at $x = c$.]

18. Let $f, g : [0, 1] \to \mathbb{R}$ such that $f(0) = g(0)$ and $f'(x) > g'(x)$ for all $x \in (0, 1)$. Show that $f(x) > g(x)$ must hold for all $x \in (0, 1]$.

19. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ for every $x \in (a, b)$, prove that f must be strictly increasing on $[a, b]$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, with positive second derivative. Prove that, $f(x + f'(x)) \geq f(x)$ for every $x \in \mathbb{R}$.

21. Show that the equation $3^x + 4^x + 5^x = 6^x$ has exactly one real root.

22. For non-zero $a_1, \ldots, a_n$ and for distinct $\theta_1, \ldots, \theta_n$, show that the equation
$$a_1 x^{\theta_1} + a_2 x^{\theta_2} + \cdots + a_n x^{\theta_n} = 0$$
has at most $n - 1$ roots for $x \in (0, \infty)$.

23. (*Cauchy's MVT*) Let f, g be continuous on $[a, b]$, differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that $(f(b) - f(a)) \cdot g'(c) = (g(b) - g(a)) \cdot f'(c)$.

    Comment: If we also assume that $g'(x) \neq 0$ for every $x \in (a, b)$, then it rearranges to the following:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

24. Suppose that f is continuous on $[a, b]$, differentiable on (a, b), where $0 < a < b$. Prove that there exists $c_1, c_2 \in (a, b)$ such that

$$\frac{f'(c_2)}{a + b} = \frac{f'(c_1)}{2c_1}.$$

25. Let f be continuous on $[a, b]$, differentiable on (a, b), where $a > 0$. Show that there exists $c \in (a, b)$ such that

$$\frac{bf(a) - af(b)}{b - a} = f(c) - cf'(c).$$

26. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), where $b - a \geq \pi$. Prove that there exists $x_0 \in (a, b)$ such that $f'(x_0) < 1 + f(x_0)^2$.

27. Suppose that $f : [a, b] \to [a, b]$ is continuous on $[a, b]$, differentiable on (a, b), and satisfies $f(a) = a, f(b) = b$. Show that there exist $c, d \in (a, b)$ such that $c \neq d$ and $f'(c)f'(d) = 1$.

28. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function. Suppose that $f(1/n) = 1$ holds for every $n \in \mathbb{N}$. Prove that $f'(0) = 0$. Furthermore, show that $f''(0) = 0$.

29. Given that, $f(x) = 8x^3 + 3x$. Find the value of $\displaystyle\lim_{x\to\infty} \frac{f^{-1}(8x) - f^{-1}(x)}{x^{1/3}}$.

30. Suppose that $\displaystyle\lim_{x\to\infty} f'(x) = \infty$. Does this imply that $f(x)$ is unbounded?

31. Let P be a polynomial with real coefficients. Let the leading coefficient be a (where $a > 0$) and the degree be b. Show that,

$$\lim_{n\to\infty} \left(P(n + 1)^{1/b} - P(n)^{1/b} \right) = a^{1/b}.$$

32. Suppose that f is twice differentiable on (a, b) with positive second derivative. Prove that for any $x, y \in [a, b]$ and any $\lambda \in [0, 1]$, it holds that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$