Some Problems on Applications of Derivatives

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November 01, 2019

1. Find maxima/minima of the following functions:
   (a) $f(x) = (x-1)^2(x-2)^2,\ x \in \mathbb{R}$, (b) $f(x) = (x-1)^2(x-2)^3,\ x \in \mathbb{R}$.

2. Prove the following inequalities:

   (a) $e^x \geq x + 1$ for every $x \in \mathbb{R}$. (This implies $\log(1+x) \leq x$ for $x > -1$.)

   (b) $1 - x^2/2! \leq \cos x$ for every $x \in \mathbb{R}$.

   (c) $x - x^3/3! \leq \sin x \leq x$ for every $x \geq 0$.

   Also observe that for each of these, equality holds if and only if $x = 0$.

3. Prove that $\left(x + \dfrac{1}{x}\right) \tan^{-1} x > 1$ holds for every $x > 0$.

4. Prove that $\log(1+x) < \dfrac{x}{\sqrt{1+x}}$ holds for every $x > 0$.

5. Show that for every $x \in \left[0, \dfrac{\pi}{2}\right]$, we have the inequality $\dfrac{\sin x}{x} \geq \dfrac{2}{\pi}$.

6. Determine which of these numbers are greater (without using any calculator): (a) e^π or π^e, (b) $2^{\sqrt{2}}$ or e.

7. Let $f(x) = |x-1| + |x-2| + |x-3| + \cdots + |x-9|,\ x \in \mathbb{R}$. Find all x that minimises $f(x)$.

8. Let $f(x) = |x-1| + |x-2| + |x-3| + \cdots + |x-10|,\ x \in \mathbb{R}$. Find all x that minimises $f(x)$.

9. Let $f(x) = (x-1)^2 + (x-2)^2 + (x-3)^2 + \cdots + (x-10)^2,\ x \in \mathbb{R}$. Find all x that minimises $f(x)$.

10. For $a, b, c > 0$ satisfying $abc = 1$, prove that $a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3$.

11. Determine the point(s) on the graph of $y = x^2 + 1$ that are closest to $(0, 2)$.

12. An open tank with square base must have the capacity of 500 litres. What should be its dimensions if least amount of building material is to be used? (i.e. minimise the surface area.)

13. A window is being built such that its bottom part is a rectangle and the top is a semi-circle. If there is 12 meters of framing materials, what must the dimensions of the window be to let in maximum possible light?

14. A rectangle $ABCD$ is inscribed in $\triangle PBQ$, which is right-angled at B, such that A, C, D lie on PB, BQ, PQ respectively. Prove that its area is maximised when D is the midpoint of PQ.

15. If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here 'cone' means a right-circular cone.)

16. Show that the semi-vertical angle of the right-circular cone that has maximum volume among all right-circular cones having a fixed slant height, say ℓ, is $\tan^{-1}\sqrt{2}$.

17. For $x \geq 2$ show that $(x+1)\cos\dfrac{\pi}{x+1} - x\cos\dfrac{\pi}{x} > 1$.

18. For a positive integer n, find all local extrema (if any) of the function

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) e^{-x},\ x \in \mathbb{R}.$$

19. Let f be continuously differentiable on $(0, \infty)$ and let $f(0) = 1$. Show that if $|f(x)| \leq e^{-x}$ holds for all $x \geq 0$, then there exists $x_0 > 0$ such that $f'(x_0) = -e^{-x_0}$.

20. Show that each of the equations

$$\sin(\cos x) = x \text{ and } \cos(\sin x) = x$$

has exactly one root in $[0, \pi/2]$. Moreover, show that if x_1 and x_2 are the roots of the former and the latter equation, respectively, then $x_1 < x_2$.

21. Find the maximum value of f on $\mathbb{R}$, where f is given by

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-1|}.$$

More problems (specially optimisation problems) can be found at the following sites:
http://tutorial.math.lamar.edu/Classes/CalcI/DerivAppsIntro.aspx.
https://www.whitman.edu/mathematics/calculus_online/section06.01.html.