Applⁿ of derivatives

13/12/2020

f is said to be increasing at $x = c$ if $\exists\, \delta > 0$ such that $f(x) \geq f(c)$ for every $x \in (c, c+\delta)$ and $f(x) \leq f(c)$ for every $x \in (c-\delta, c)$.

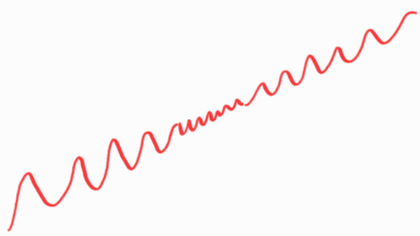
Suppose f is diff'ble at $x = c$. Then f is increasing at $x = c \iff f'(c) \geq 0$.

However,

f is increasing in a neighborhood of c $\overset{\Longrightarrow}{\not\Longleftarrow}$ $f'(c) \geq 0$.

$\because \dfrac{f(x) - f(c)}{x - c} \geq 0$ in $(c-\delta, c+\delta)$

counter-example:
(for $\Leftarrow$)



$f'(x) > 0$ in $(a, b) \Rightarrow f$ is inc on $[a, b]$

Similarly, we define "decreasing at a point". We also discussed about local maxima and minima.

How to find local maxima/minima

f diff'ble on (a, b), cont. on $[a, b]$, then

f attains a local maximum/minimum at $x = c$ <u>only if</u> $f'(c) = 0$.

Suppose $f'(c) = 0$. How to ensure that f indeed attains local max/min at $x = c$?

<u>First derivative criterion</u>

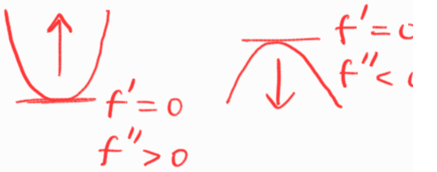
If $f'(x)$ changes its sign from $-$ve to $+$ve at $x = c$, i.e.,

$$f'(x) \gtreqless 0 \iff x \gtreqless c \text{ holds}$$

near $x = c$, then we can say that $f(x)$ attains a local min at $x = c$.

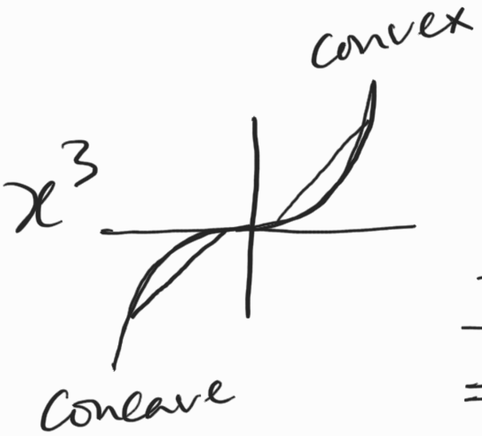
Similarly, if f' changes its sign from $+$ve to $-$ve then local max.

Second Derivative Criterion



$f'(c)=0,\ f''(c)<0 \Rightarrow$ local max at $x=c$

$f'(c)=0,\ f''(c)>0 \Rightarrow$ local min at $x=c$



$f(x)=x^3$

$f''(0)=0,\ f'''(0)\neq 0$

$\Rightarrow$ f has an inflection point at $x=c$.

In general,

Suppose $f'(c)=f''(c)=\cdots=f^{(n)}(c)=0$,

and $f^{(n+1)}(c)\neq 0$.

If n is odd, then

$f^{(n+1)}(c)>0 \Rightarrow f$ has local min at c

$f^{(n+1)}(c)<0 \Rightarrow f$ has local max at c

If n is even, then

f has an inflection point at $x=c$

One fact we missed earlier:

Suppose f' exists on $(c-\delta, c+\delta)\setminus\{c\}$, and f is cont. at c as well.

If $\lim\limits_{x\to c} f'(x) = l$, then $f'(c) = l$.

Proof

For any $x > c$, apply MVT to f on $[c, x]$ to write

$$\frac{f(x)-f(c)}{x-c} = f'(t_x) \quad \text{---} (*)$$

where $c < t_x < x$. When $x \to c$, we get $t_x \to c$, hence the RHS of $(*)$ tends to l. Handle the case $x \to c^-$ similarly.

$$f(x) = \begin{cases} \cdots & \text{for } x > c \\ \cdots & \text{for } x = c \\ \cdots & \text{for } x < c \end{cases}$$

Easier to calculate $f'(x)$ for $x > c$ and $x < c$. Then, if

$$\lim_{x\to c} f'(x)$$

exists, then it will be same as $f'(c)$.