Fermat's theorem and the Mean Value Theorems

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Last updated: Dec 09, 2020

1. Suppose that $f : [a, b] \to \mathbb{R}$ has a local maximum at $x = c$, where $c \in (a, b)$. Is it necessary that $f'(c) = 0$?

2. Suppose that $f : [a, b] \to \mathbb{R}$ has a local maximum at $x = c$, where $c \in [a, b]$. Assume that f is differentiable at $x = c$. Is it necessary that $f'(c) = 0$?

3. (*Fermat's Theorem*) Suppose that $f : [a, b] \to \mathbb{R}$ has a local maximum (or minimum) at $x = c$, where $c \in (a, b)$. Assume that f is differentiable at $x = c$. Show that $f'(c)$ must be equal to zero.

   (Note: It is enough to show it only for the case of local maximum. If f has a local minimum at $x = c$, then we can just apply that result to $-f$.)

4. Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable at $x = c$, where $c \in (a, b)$. If $f'(c) = 0$, is it necessary that f has a local maximum/minimum at $x = c$?

5. (*Rolle's Theorem*) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Prove that there exists $c \in (a, b)$ such that $f'(c) = 0$.

6. Show that Rolle's theorem fails to hold if f is not continuous at one of endpoints (a or b).

7. (*Lagrange's Mean Value Theorem*) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

8. Let f be continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$. Prove that for any $\beta \in \mathbb{R}$, there exists some $c \in (a, b)$ such that $f'(c) + \beta \cdot f(c) = 0$.

9. Let f, g be continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$. Show that there exists $c \in (a, b)$ such that $g'(x)f(x) + f'(x) = 0$.

10. Let f be continuous on $[0, \pi]$, differentiable on $(0, \pi)$. Show that there exists $c \in (0, \pi)$ such that $f'(c) \sin c + f(c) \cos c = 0$.

11. Assume that f is continuous on $[a, b]$, differentiable on (a, b), where $0 < a < b$, and suppose that $bf(a) = af(b)$. Prove that there exists $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

12. Suppose that f is continuous on $[a, b]$, differentiable on (a, b), and suppose that $f(b)^2 - f(a)^2 = b^2 - a^2$. Prove that the equation $f'(x)f(x) = x$ has at least one root in (a, b).

13. Let f and g be continuous and never vanishing on $[a, b]$ and differentiable on (a, b). Prove that if $f(a)g(b) = f(b)g(a)$ then there is $x_0 \in (a, b)$ such that

$$\frac{f'(x_0)}{f(x_0)} = \frac{g'(x_0)}{g(x_0)}.$$

14. Assume that $a_0, a_1, \ldots, a_n$ are real numbers such that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Show that $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ has at least one root in $(0, 1)$.

15. Let $P(x)$ be a polynomial with real coefficients and degree $n \geq 2$. Prove that if all roots of $P(x)$ are real, then all roots of $P'(x)$ are also real.

16. Let f be continuous on $[0, 2]$ and twice differentiable on $(0, 2)$. If $f(x) = x$ holds for $x = 0, 1$ and 2, then show that there exists $x_0 \in (0, 2)$ such that $f''(x_0) = 0$.

17. Let $n > 1$ be an integer, and let $f : [a, b] \to \mathbb{R}$ be a continuous function, which is n-times differentiable on (a, b). If the graph of $f(x)$ has $n + 1$ collinear points, then prove that there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$.
[Here $f^{(n)}(c)$ denotes the n-th derivative of $f(x)$, evaluated at $x = c$.]

18. Let $f, g : [0, 1] \to \mathbb{R}$ such that $f(0) = g(0)$ and $f'(x) > g'(x)$ for all $x \in (0, 1)$. Show that $f(x) > g(x)$ must hold for all $x \in (0, 1]$.

19. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ for every $x \in (a, b)$, prove that f must be strictly increasing on $[a, b]$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, with positive second derivative. Prove that, $f(x + f'(x)) \geq f(x)$ for every $x \in \mathbb{R}$.

21. Show that the equation $3^x + 4^x + 5^x = 6^x$ has exactly one real root.

22. For non-zero $a_1, \ldots, a_n$ and for distinct $\theta_1, \ldots, \theta_n$, show that the equation

$$a_1 x^{\theta_1} + a_2 x^{\theta_2} + \cdots + a_n x^{\theta_n} = 0$$

has at most $n - 1$ roots for $x \in (0, \infty)$.

23. (*Cauchy's MVT*) Let f, g be continuous on $[a, b]$, differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that $(f(b) - f(a)) \cdot g'(c) = (g(b) - g(a)) \cdot f'(c)$.

    Comment: If we also assume that $g'(x) \neq 0$ for every $x \in (a, b)$, then it rearranges to the following:
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

24. Suppose that f is continuous on $[a, b]$, differentiable on (a, b), where $0 < a < b$. Prove that there exists $c_1, c_2 \in (a, b)$ such that
$$\frac{f'(c_2)}{a+b} = \frac{f'(c_1)}{2c_1}.$$

25. Let f be continuous on $[a, b]$, differentiable on (a, b), where $a > 0$. Show that there exists $c \in (a, b)$ such that
$$\frac{bf(a) - af(b)}{b - a} = f(c) - cf'(c).$$

26. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), where $b - a \geq \pi$. Prove that there exists $x_0 \in (a, b)$ such that $f'(x_0) < 1 + f(x_0)^2$.

27. Suppose that $f : [a, b] \to [a, b]$ is continuous on $[a, b]$, differentiable on (a, b), and satisfies $f(a) = a, f(b) = b$. Show that there exist $c, d \in (a, b)$ such that $c \neq d$ and $f'(c)f'(d) = 1$.

28. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function. Suppose that $f(1/n) = 1$ holds for every $n \in \mathbb{N}$. Prove that $f'(0) = 0$. Furthermore, show that $f''(0) = 0$.

29. Given that, $f(x) = 8x^3 + 3x$. Find the value of $\displaystyle\lim_{x\to\infty} \frac{f^{-1}(8x) - f^{-1}(x)}{x^{1/3}}$.

30. Suppose that $\displaystyle\lim_{x\to\infty} f'(x) = \infty$. Does this imply that $f(x)$ is unbounded?

31. Let P be a polynomial with real coefficients. Let the leading coefficient be a (where $a > 0$) and the degree be b. Show that,
$$\lim_{n\to\infty} \left(P(n+1)^{1/b} - P(n)^{1/b}\right) = a^{1/b}.$$

32. Suppose that f is twice differentiable on (a, b) with positive second derivative. Prove that for any $x, y \in [a, b]$ and any $\lambda \in [0, 1]$, it holds that
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Solutions

1. No, because f might not be differentiable at c. Example: $f : [-1, 1] \to \mathbb{R}$, $f(x) = -|x|$. f has a local maximum at 0, but $f'(0)$ does not exist.

2. No. For example, take $f : [0, 1] \to \mathbb{R}$, $f(x) = x$ has a local maximum at $x = 1$, but $f'(1) = 1$. (Note, here $f'(1)$ is defined as the left-hand derivative, because f is not defined on the right side of $x = 1$.)

3. Let $f : [a, b] \to \mathbb{R}$ have a local maximum at $x = c$, where $c \in (a, b)$, and $f'(c)$ exists. From definition of local maximum, there exists $\delta > 0$ such that $f(x) \leq f(c)$ holds for every $x \in (c - \delta, c + \delta)$. So, for every $x \in (c - \delta, c)$, we have $\dfrac{f(x) - f(c)}{x - c} \geq 0$, which implies that

$$f'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

   On the other hand, for every $x \in (c, c + \delta)$, we have $\dfrac{f(x) - f(c)}{x - c} \leq 0$, which implies that

$$f'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \leq 0.$$

   Combining the above two, we conclude that $f'(c) = 0$.

4. No. For example, take $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^3$. f does not have a local maximum or a local minimum at $x = 0$ (can you see why?), although $f'(0) = 0$.

5. Since f is continuous on $[a, b]$, the *Extreme Value Theorem* tells us that f attains a maximum and a minimum value inside the interval $[a, b]$. Suppose f attains its maximum at c_1 and its minimum at c_2. If c_1 lies inside (a, b), then we can apply Fermat's theorem to conclude that $f'(c_1) = 0$. Similarly, if c_2 lies inside (a, b), then we can conclude that $f'(c_2) = 0$. If none among c_1, c_2 lies inside (a, b) then c_1, c_2 are essentially the endpoints. In this case, the maximum of f coincides with its minimum (since $f(a) = f(b)$), hence f must be constant throughout $[a, b]$. Thus, we conclude that in all the cases, there exists $c \in (a, b)$ such that $f'(c) = 0$.

6. Take $[a, b] = [0, 1]$ and define $f(x) = x$ for $0 \leq x < 1$ and set $f(1) = 0$.

7. I usually illustrate the derivation of Lagrange's MVT using a picture like Figure 1. If we tilt our head and view the picture in such a way that the blue lines look like the horizontal and the vertical axis, then we see that we can apply Rolle's theorem *in that picture* to conclude that at some point on the curve, *the tangent should be parallel to the blue x-axis.* Then, tilting our head back to the old position, we can see that what we argued just now,

essentially means that at some point on the curve, the tangent should be parallel to the blue line joining $(a, f(a))$ and $(b, f(b))$.

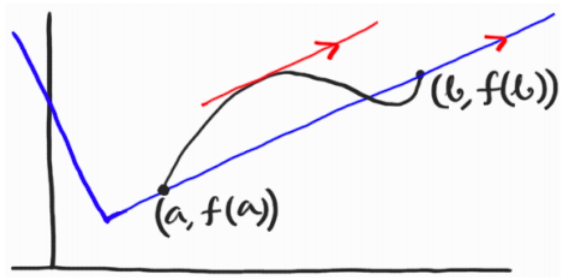

Figure 1: Deriving LMVT from Rolle's theorem

If you want a more mathematical proof, just subtract from $f(x)$ the equation of the blue line joining $(a, f(a))$ and $(b, f(b))$ and apply Rolle's theorem to that new function. To be precise, apply Rolle's theorem to the function

$$g(x) = f(x) - \ell(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right)$$

on the interval $[a, b]$.

8. Apply Rolle's theorem to $h(x) = e^{\beta x} f(x)$.

   A stupid-but-useful method for such problems:

   We wish to apply MVT or Rolle's theorem on a special function $h(\cdot)$ such that $h'(c)$ gives the desired expression. A similar idea appears in the study of first order linear differential equations. Here we want $f'(c) + \beta f(c) = 0$ for some c. Let us instead pretend that it holds for all c, and try to figure out what that means. Writing $y = f(x)$ (for simplicity),

$$\begin{aligned} y' + \beta y = 0 &\implies y'/y = -\beta \\ &\implies \log y = -\beta x + c \qquad \text{(by integration)} \\ &\implies y = e^{-\beta x + c} \implies y e^{\beta x} = \text{ constant.} \end{aligned}$$

   Once we arrived at $e^{\beta x} f(x) =$ constant, be assured that this is the required function (on which we need to apply Rolle's theorem/MVT).

   Note, the above calculation is just a part of our rough work. The solution always starts with saying that "Let us consider the function $h(x) = e^{\beta x} f(x) \ldots$".

   Though this method looks stupid, I found that it often helps, at least to get a function to start with. Once you learn the motivation behind a similar trick used to solve first order linear differential equations, you will understand why this method works.

9. (Rough work: Here we have $g'(x)y + y' = 0$. Solving this, we get $y = c \cdot e^{-g(x)}$ i.e. $e^{g(x)} f(x) = \text{constant}$.) Consider $h(x) = e^{g(x)} f(x)$. Apply Rolle's theorem to this function on the interval $[a, b]$.

10. Apply Rolle's theorem to $h(x) = f(x) \sin x$. (Even if you did not figure it out using the *stupid method*, I urge you to do that, just for the sake of illustration.)

11. Apply Rolle's theorem to $h(x) = f(x)/x$. This is somewhat evident from the given condition (which can be rewritten as: $f(a)/a = f(b)/b$).

12. Apply Rolle's theorem to $h(x) = f(x)^2 - x^2$.

13. Apply Rolle's theorem to $h(x) = f(x)/g(x)$.

14. Consider the function

$$Q(x) = a_0 x + a_1 \frac{x^2}{2} + \cdots + a_n \frac{x^n}{n+1},$$

which is actually $\int_0^x P(x)dx$. Observe that $P(x) = Q'(x)$ and we are given that $Q(0) = Q(1) = 0$. We get the desired result by applying Rolle's theorem to $Q(x)$.

15. Let $P(x) = a(x - r_1)^{c_1}(x - r_2)^{c_2} \cdots (x - r_k)^{c_k}$ where $r_1, r_2, \ldots, r_k$ are the distinct roots of P, with multiplicities $c_1, c_2, \ldots, c_k$ respectively. Here $c_1, c_2, \ldots, c_k$ are natural numbers, which sum up to n, the degree of P. Observe that,

(i) In each (r_i, r_{i+1}) there is a root of $P'(x)$, in light of Rolle's theorem.

(ii) The root r_i of $P(x)$ is also a root of $P'(x)$, with multiplicity $c_i - 1$.

Thus, the number of roots of $P'(x)$, counted with multiplicities, is at least

$$\underbrace{(k-1)}_{\text{coming from (i)}} + \underbrace{\sum_{j=1}^{k} (c_j - 1)}_{\text{coming from (ii)}} = k - 1 + n - k = n - 1.$$

Since $P'(x)$ cannot have more than $n - 1$ roots, we conclude that all the roots of $P'(x)$ are real. (Comment: here we should have used 'zero(s)' in place of 'root(s)'.)

16. Applying LMVT to f on $[0, 1]$ and on $[1, 2]$, we get $c_1 \in (0, 1)$ and $c_2 \in (1, 2)$ such that

$$f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = 1, \quad f'(c_2) = \frac{f(2) - f(1)}{2 - 1} = 1.$$

Now we apply Rolle's theorem to f', on the interval $[c_1, c_2]$, to get the desired x_0.

17. Call those collinear points to be (x_i, y_i), $1 \leq i \leq n+1$. Now, for each $1 \leq i \leq n$, apply LMVT to f on the interval $[x_i, x_{i+1}]$ to get one $c_i \in (x_i, x_{i+1})$ such that $f'(c_1) = f'(c_2) = \cdots = f'(c_n) =$ slope of the line that contains those $n+1$ points. Next, apply Rolle's theorem on f' to get hold of $f''(d_1) = f''(d_2) = \cdots = f''(d_{n-1}) = 0$. Now apply Rolle's theorem on f'' ... I think you got the idea.

18. Let, if possible, $f(b) \leq g(b)$ for some $0 < b \leq 1$. Then, applying LMVT to $h(x) = g(x) - f(x)$, we obtain that there exists $c \in (0, b)$ such that

$$h'(c) = \frac{h(b) - h(0)}{b - 0} = \frac{g(b) - f(b)}{b} \geq 0.$$

But $h'(c) = g'(c) - f'(c) < 0$. Hence we get a contradiction.

19. Take any $x, y \in [a, b]$, $x < y$ (say) and apply MVT to f on the interval $[x, y]$. This result is really important, and will be used often in the rest of problems (or even in the course!).

20. If x is such that $f'(x) = 0$, then the given inequality holds with equality. If for some x, $f'(x) < 0$, then MVT applied to f on the interval $[x + f'(x), x]$ yields that $f(x) - f(x + f'(x)) = f'(c)(-f'(x))$, for some c with $x + f'(x) < c < x$. Now, since the second derivative is positive, f' is increasing. Hence $c < x \implies f'(c) < f'(x) < 0$. Therefore, $f(x) - f(x + f'(x)) < 0$, which yields the required inequality. In the case $f'(x) > 0$, by the same argument we have $f(x + f'(x)) - f(x) = f'(x)f'(c)$, for some c between x and $x + f'(x)$. And we have $x < c \implies f'(c) > f'(x) > 0$. Hence we get $f(x) - f(x + f'(x)) < 0$, as desired.

21. It is easily seen that $x = 3$ is a root of the equation. Consider the function $f(x) = (3/6)^x + (4/6)^x + (5/6)^x$. If there be some other root of the equation, then Rolle's theorem would imply that $f'(c) = 0$ for some $c \in \mathbb{R}$. But we observe that, $f'(x) < 0$ for every $x \in \mathbb{R}$. Another solution can be sketched out using only the fact that f is decreasing.

22. Use induction on n. The claim is trivial for $n = 1$. Suppose it holds for $n = k$. Then, for $n = k+1$, rewrite the equation as

$$f(x) = a_1 + a_2 x^{\theta_2 - \theta_1} + a_3 x^{\theta_3 - \theta_1} \cdots + a_n x^{\theta_n - \theta_1} = 0.$$

If this equation has more than $n - 1 = k$ roots, then by Rolle's theorem, the equation $f'(x) = 0$ will have more than $k-1$ roots. But this contradicts the induction hypothesis for $n = k$ (since the equation $f'(x) = 0$ is of the same form). (Note: The induction hypothesis should be: for any non-zero $a_1, a_2, \ldots, a_n$ and any distinct $\theta_1, \ldots, \theta_n$, the equation $a_1 x^{\theta_1} + \cdots + a_n x^{\theta_n} = 0$ has at most $n-1$ roots in $(0, \infty)$.)

23. Apply Rolle's theorem on the function

$$h(x) = \det \begin{pmatrix} 1 & f(x) & g(x) \\ 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \end{pmatrix}.$$

Note that $h(x)$ is just a linear combination of $f(x)$ and $g(x)$, i.e. we can write $h(x) = k + m \cdot f(x) + n \cdot g(x)$ for some constants k, m, n.

24. Applying Cauchy's MVT with the functions $f(x)$ and $g(x) = x^2$, we can say that there exists $c_1 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c_1)}{2c_1}. \tag{$*$}$$

On the other hand, LMVT applied to $f(x)$ provides $c_2 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c_2).$$

Substituting this in $(*)$, we get the desired result.

25. Applying Cauchy's MVT to the functions $f(x)/x$ and $1/x$ on the interval $[a, b]$, we can say that there exists $c \in (a, b)$, for which

$$\frac{f(b)/b - f(a)/a}{1/b - 1/a} = \frac{(cf'(c) - f(c))/c^2}{-1/c^2}.$$

After a little manipulation, this is seen to be same as the desired equation.

26. Let, if possible, $f'(x) \geq 1 + f(x)^2$ for every $x \in (a, b)$. Consider $g(x) = \tan^{-1} f(x)$. (Motivation? Because $\int \frac{f'(x)}{1+f(x)^2} dx = \tan^{-1} f(x) + c$.) We apply LMVT on g in the interval $[a, b]$ to arrive at

$$\frac{g(b) - g(a)}{b - a} = g'(c) = \frac{f'(c)}{1 + f(c)^2} \geq 1.$$

This implies that $g(b) - g(a) \geq b - a \geq \pi$. But, on the other hand we have

$$g(b) - g(a) = \tan^{-1} f(b) - \tan^{-1} f(a) < \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi.$$

27. Apply LMVT with the function $g(x) = f(f(x))$ to get $x_0 \in (a, b)$ such that

$$g'(x_0) = 1 \implies f'(f(x_0))f'(x_0) = 1.$$

Now, if $f(x_0) \neq x_0$ then we take $c = x_0, d = f(x_0)$ and we are through.

Else, we have $f(a) = a, f(x_0) = x_0, f(b) = b$. Apply LMVT to f on the intervals $[a, x_0]$ and $[x_0, b]$ to get $c \in (a, x_0)$ and $d \in (x_0, b)$ such that

$$f'(c) = \frac{f(x_0) - f(a)}{x_0 - a} = 1, \text{ and } f'(d) = \frac{f(b) - f(x_0)}{b - x_0} = 1.$$

28. First note that $f(0) = \lim_{n\to\infty} f(1/n) = 1$. Next, since $f'(0)$ exists, we can write

$$f'(0) = \lim_{n\to\infty} \frac{f(1/n) - f(0)}{1/n - 0} = 0.$$

Because if we know that $\lim_{x\to a} g(x)$ exists, then we can evaluate the limit using any particular sequence. Since we also know that $f''(0)$ exists, we shall evaluate it using one particular sequence. The choice of this sequence is going to be very special. For each $n \geq 1$, we apply LMVT to f on the interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ to get

$$f'(c_n) = \frac{f(\frac{1}{n}) - f(\frac{1}{n+1})}{\frac{1}{n} - \frac{1}{n+1}} = 0 \text{ for some } c_n, \text{ where } \frac{1}{n+1} < c_n < \frac{1}{n}.$$

Now, invoking Sandwich theorem, we can say that $c_n \to 0$ as $n \to \infty$. Hence,

$$f''(0) = \lim_{n\to\infty} \frac{f'(c_n) - f'(0)}{c_n - 0} = \lim_{n\to\infty} \frac{0 - 0}{c_n - 0} = 0.$$

29. Call $f^{-1} = g$ and $h(x) = x^{1/3}$. Cauchy's MVT gives

$$\frac{f^{-1}(8x) - f^{-1}(x)}{x^{1/3}} = \frac{g(8x) - g(x)}{h(8x) - h(x)} = \frac{g'(c)}{h'(c)}$$

for some $c \in (x, 8x)$. Observe that,

$$\frac{g'(c)}{h'(c)} = \frac{3c^{2/3}}{f'(f^{-1}(c))} = \frac{3c^{2/3}}{24f^{-1}(c)^2 + 3}.$$

Substituting $d = f^{-1}(c)$ here, we obtain

$$\frac{f^{-1}(8x) - f^{-1}(x)}{x^{1/3}} = \frac{3(8d^3 + 3d)^{2/3}}{24d^2 + 3}.$$

Now, we let $x \to \infty$ (which implies $c \to \infty$ and hence $d \to \infty$) to write

$$\lim_{x\to\infty} \frac{f^{-1}(8x) - f^{-1}(x)}{x^{1/3}} = \lim_{d\to\infty} \frac{3(8d^3 + 3d)^{2/3}}{24d^2 + 3} = \frac{1}{2}.$$

30. Yes, we can conclude that f must be unbounded.

Proof: Let, if possible, f be bounded. Then there exists $M > 0$ such that $|f(x)| < M$ holds for all x. Now, for every positive integer n, we can apply MVT to f in the interval $[n, n+1]$ to get

$$f(n+1) - f(n) = f'(c_n),$$

for some $c_n \in (n, n+1)$. Notice that $c_n \to \infty$ as $n \to \infty$. Hence we must have

$$\lim_{n\to\infty} f'(c_n) = \infty. \tag{$\star$}$$

But, for each n, $|f'(c_n)| = |f(n+1) - f(n)| \leq 2M$. This shows that $f'(c_n)$ is a bounded sequence, which contradicts ($\star$).

31. Consider $f(x) = P(x)/a$. Then $f(x)$ is a monic polynomial with real coefficients and has degree b. It suffices to show that

$$\lim_{x\to\infty} \left(f(x+1)^{1/b} - f(x)^{1/b} \right) = 1. \tag{1}$$

Note, since the leading coefficient of f is positive, so $f(x) > 0$ for all $x >$ some x_0. Hence, $f(x)^{1/b}$ is well-defined for $x > x_0$. For any fixed $x > x_0$, Mean Value Theorem implies that there exists c between $(x, x+1)$ such that

$$\frac{f(x+1)^{1/b} - f(x)^{1/b}}{x+1-x} = \frac{1}{b} f(c)^{\frac{1}{b}-1} f'(c). \tag{2}$$

Here c depends on x. But the only fact we care about is that $x < c < x+1$. Thus, $c \to \infty$ if $x \to \infty$. Now, in view of the equations (1) and (2), it suffices to show that

$$\lim_{t\to\infty} \frac{1}{b} f(t)^{1-\frac{1}{b}} f'(t) = 1. \tag{3}$$

Proving the last limit is easy. Think of writing $f(t)$ as $t^b + a_{b-1}t^{b-1} + \cdots + a_1 t + a_0$. Then,

$$\lim_{t\to\infty} \frac{f'(t)}{b\ f(t)^{\frac{b-1}{b}}} = \lim_{t\to\infty} \frac{bt^{b-1} + (\text{a poly. of degree} \leq b-2)}{b\Big(t^b + \text{a poly. of degree} \leq b-1\Big)^{\frac{b-1}{b}}}.$$

Dividing the numerator and denominator by t^{b-1}, the last limit equals

$$= \lim_{t\to\infty} \frac{b + \text{some } 1/t, 1/t^2 \text{ etc.}}{b\Big(1 + \text{some } 1/t, 1/t^2 \text{ etc.}\Big)^{\frac{b-1}{b}}} = 1.$$

32. Fix any $x, y \in [a, b]$ and $\lambda \in (0, 1)$ (if $\lambda = 0$ or 1 then the conclusion is trivial). Call $\lambda x + (1 - \lambda)y = z$. Then, $x < z < y$ and observe that

$$\lambda x + (1-\lambda)y = z \iff \lambda = \frac{y-z}{y-x}.$$

Now, the inequality that we want to show, can be written as

$$f(z) \leq \frac{y-z}{y-x} f(x) + \frac{z-x}{y-x} f(y). \tag{1}$$

We can rearrange this to write

$$\frac{y-z}{y-x}(f(z) - f(x)) \leq \frac{z-x}{y-x}(f(y) - f(z)),$$

or,

$$\frac{f(z) - f(x)}{z-x} \leq \frac{f(y) - f(z)}{y-z}. \tag{2}$$

This can be easily proved, using LMVT. We apply LMVT to f on the intervals $[x, z]$ and $[z, y]$ separately to get

$$\frac{f(z) - f(x)}{z-x} = f'(c_1) \text{ and } \frac{f(y) - f(z)}{y-z} = f'(c_2) \tag{3}$$

for some c_1, c_2 such that $c_1 \in (x, z)$ and $c_2 \in (z, y)$. Now, since $f'' > 0$ implies that f' is increasing, we have $c_1 < z < c_2 \implies f'(c_1) \leq f'(c_2)$. This, combined with (3), proves (2).

Comment: The inequality $f'(c_1) \leq f'(c_2)$ is actually strict, because $f'' > 0$ on (a, b). Hence, the required inequality is actually a strict inequality, if f'' happens to be strictly positive on (a, b).