Ramanujan School of Mathematics

Class Test 3 on Calculus

Time allotted: 2 hours Total points: 40

Attempt all the questions. You can use any result discussed in the class, but you have to state it properly. Since it is a 'take-home' exam, I can only request you to take the test honestly and abide by the time limit. Do not cheat to yourself. All the best!

1. (5+5 points)

   (a) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that $g(x_0) = 0$ and $g'(x) > g(x)$ for every $x \in \mathbb{R}$. Prove that $g(x) > 0$ for every $x > x_0$.

   (b) For any $a > 0$ prove that the equation $ae^x = 1 + x + x^2/2$ has a unique real root.

2. (3+3+4 points) Let $p_1, p_2, \ldots, p_n \in (0, 1)$ such that $\sum_{i=1}^n p_i = 1$. For $\alpha \geq 0$, $\alpha \neq 1$, define
$$\mathrm{H}(\alpha) = \frac{1}{1-\alpha} \log\left(\sum_{i=1}^n p_i^\alpha\right).$$

   (a) Calculate $\lim_{\alpha \to 1} \mathrm{H}(\alpha)$.

   (b) Show that
$$\frac{d\mathrm{H}(\alpha)}{d\alpha} = \frac{1}{(1-\alpha)^2} \sum_{i=1}^n z_i \log(p_i/z_i),$$
   where $z_i = p_i^\alpha / \sum_{j=1}^n p_j^\alpha$.

   (c) Hence or otherwise show that $\mathrm{H}(\alpha)$ is a non-increasing function of α.

3. (10 points) Find the area of the largest circle centred at the origin which is inscribed between the graphs of the functions
$$y = \frac{1}{1+x^2}, \text{ and } y = -\frac{1}{1+x^2}.$$
   (You need not find an approximate value, just find an expression.)

4. (10 points) Suppose that $f : [0, \pi] \to \mathbb{R}$ is a differentiable function (assume that the one sided derivatives exist at both the endpoints) such that there is no $x \in [0, \pi]$ satisfying
$$f(x) = f'(x) = 0.$$
   Show that the set $S = \{x : f(x) = 0\}$ must be finite.