Convexity 28/02/21

Warm up:

If f is twice diff'ble at c, then

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2}$$

will be equal to $f''(c)$. However, the existence of the above limit does not imply that f is twice diff'ble at c.

Sol^n For the second part, we take $c=0$, and then the above limit will exist for any odd function f, which need not be diff'ble at 0, e.g.,
$$f(x) = \begin{cases} x\cos\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For the first part, we apply L'Hôpital's rule: we have $0/0$ form, derivative of denom. does not vanish anywhere except 0, and we'll see that limit of $\frac{(\text{num})'}{(\text{denom})'}$ exists.

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2}$$

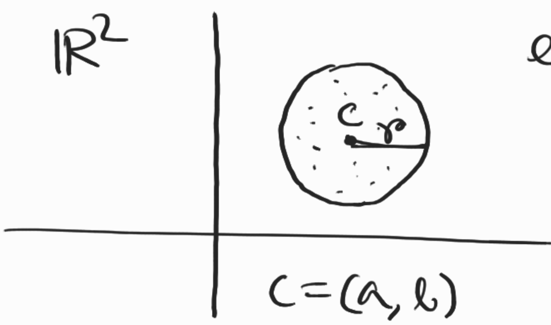
$$= \lim_{h\to 0} \frac{f'(c+h)-f'(c-h)}{2h} \quad \text{(diff. w.r.t. } h)$$

$$= \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{2h} + \frac{f'(c) - f'(c-h)}{2h}$$

$$= \frac{f''(c)}{2} + \frac{f''(c)}{2} = f''(c). \qquad \square$$

Convex functions

Convex sets



$S \subset \mathbb{R}^2$ is convex if for every x, y in S and all $\lambda \in [0,1]$, it holds that $\lambda x + (1-\lambda) y \in S$.

Is the disc

$$\{(x,y) : (x-a)^2 + (y-b)^2 \leq r^2\} \text{ convex?}$$

Pick any $(x_1, y_1), (x_2, y_2)$ in S and take any $\lambda \in [0,1]$. $(x_3, y_3) = \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)$

$$(x_3 - a, y_3 - b) = \underline{\lambda(x_1 - a, y_1 - b)} + \underline{(1-\lambda)(x_2 - a, y_2 - b)}$$

$z_j = (x_j - a, y_j - b)$

$$\|z_j\| = \sqrt{(x_j - a)^2 + (y_j - b)^2}$$

$$= \underline{\text{modulus of } (x_j - a) + i(y_j - b)}$$

$$\|z_3\| = \|\lambda z_1 + (1-\lambda) z_2\|$$

$$\leq \|\lambda z_1\| + \|(1-\lambda) z_2\| \quad \text{(triangle inequality for modulus of complex no.)}$$

$$= \lambda \|z_1\| + (1-\lambda)\|z_2\|$$

$$\leq \lambda r + (1-\lambda) r = r.$$

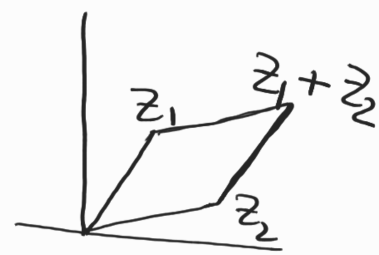
Another way:

For $z = (x, y)$ in $\mathbb{R}^2$, define

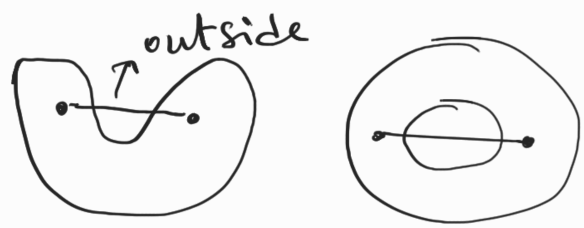
$$\|z\| = \sqrt{x^2 + y^2}.$$

Show that for z_1, z_2 in $\mathbb{R}^2$,

$$\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|.$$



Non-convex set:



Convex sets in $\mathbb{R}$: Intervals, singleton sets $[a, a]$

Let X be a convex subset of $\mathbb{R}^d$ $(d \geq 1)$.

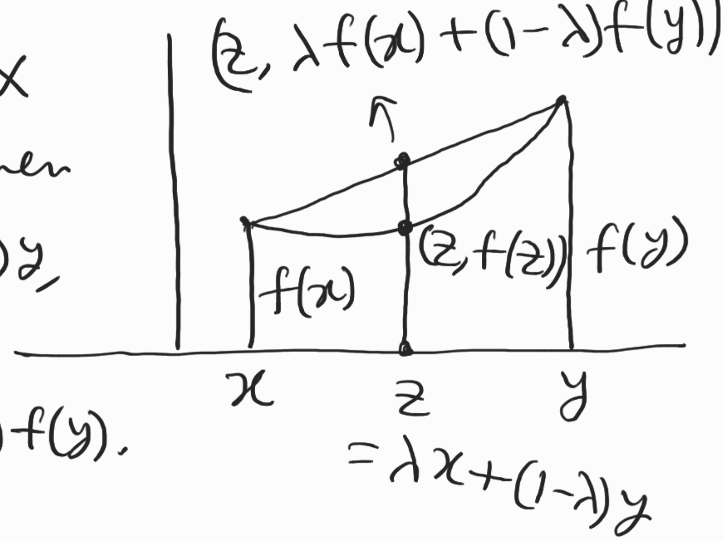
$f: X \to \mathbb{R}$ is convex if for every $x, y \in X$, and any $\lambda \in [0, 1]$, it holds that

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

$X \subseteq \mathbb{R}$, $X \to$ an interval.

Take any $x, y \in X$ and $\lambda \in [0,1]$. Then for $z = \lambda x + (1-\lambda) y$,

$f(z)$
$\leq \lambda f(x) + (1-\lambda) f(y)$.



When $\leq$ is replaced by $\geq$ in the above definition, we say that f is concave.

Thus, f is concave iff $-f$ is convex.

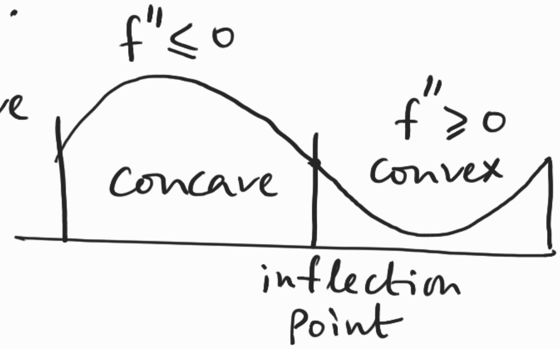
<u>Examples</u>

$|x|$ convex? By the triangle ineq.

x^2, x^4 convex? Pictorially obvious.

<u>Problem</u> $f : X \to \mathbb{R}$, X open interval. f twice diff'ble on X. Show that f is convex iff $f'' \geq 0$ on X.

Similarly, f is concave iff $f'' \leq 0$ on X.



A bad function: $f(x) = \int_0^x t^2 \sin\frac{1}{t}\, dt$

$f'(0) = 0$ $f''(0) = 0$ $f''' \to$ does not exist.

f does not have a local min/max or inflection point at 0.

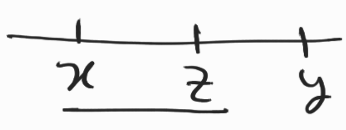
Sol^n (to the last problem)

$f'' \geq 0$ on $X \Rightarrow f$ convex on X.

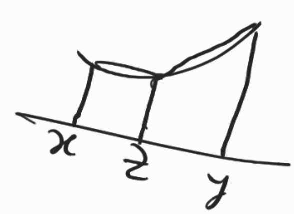
Fix any $x, y \in X$, $\lambda \in [0, 1]$. We've to show that for $z = \lambda x + (1-\lambda) y$,

$$f(z) \leq \lambda f(x) + (1-\lambda) f(y). \qquad (*)$$

Note, $z = \lambda x + (1-\lambda) y$ holds iff



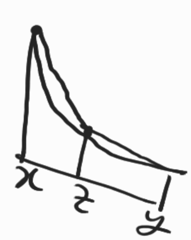
$$\lambda = \frac{y-z}{y-x}.$$



$\therefore$ $(*)$ is equivalent to,

$$f(z) \leq \frac{y-z}{y-x} f(x) + \frac{z-x}{y-x} f(y)$$

or, $$\frac{f(z) - f(y)}{z - y} \geq \frac{f(z) - f(x)}{z - x}.$$



By MVT,
$$\frac{f(z)-f(y)}{z-y} = f'(c_2), \quad c_2 \in (z,y),$$
and
$$\frac{f(z)-f(x)}{z-x} = f'(c_1), \quad c_1 \in (x,z).$$

Now, $f'' \geq 0 \Rightarrow f'$ is increasing,

so
$$c_1 < z < c_2 \Rightarrow f'(c_2) \geq f'(c_1),$$

which completes the proof.

Next we show that
$$\begin{array}{l} f \text{ convex on } X \\ \text{and } f'' \text{ exists} \end{array} \Rightarrow f'' \geq 0 \text{ on } X.$$

Take any $c \in X$. Pick $h > 0$ small enough so that $c-h, c+h \in X$. Then by the convexity of f,
$$f\left(\frac{c-h+c+h}{2}\right) \leq \frac{f(c-h)+f(c+h)}{2}$$
$$\Rightarrow \quad \frac{f(c+h)+f(c-h)-2f(c)}{h^2} \geq 0$$
for all small $h > 0$. Taking limit as $h \to 0^+$, we get $f''(c) \geq 0$.

Try to prove the above without using the problem we did as warm-up today.

Now we can find a variety of convex or concave functions.

Examples

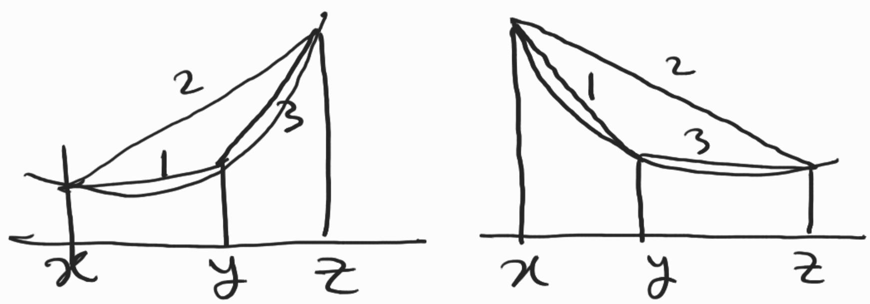
- x^2, x^4 are convex on $\mathbb{R}$.
- $\sin x$ is concave on $[0, \pi]$.
- $\cos x$ is concave on $[0, \pi/2]$.
- $e^x, e^{-x}, e^{\alpha x}$ convex on $\mathbb{R}$.
- $\log x, \sqrt{x}$ concave on $(0, \infty)$.
- $1/x$ convex on $(0, \infty)$, concave on $(-\infty, 0)$.
- x^3, x^5 are convex on $(0, \infty)$, but concave on $(-\infty, 0)$. For them, $x = 0$ is the inflection point.
- Example of non-cont. convex fn?

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 0, 1 \end{cases}$$ discont. at end points

Check that it is convex.

Problem $f: X \to \mathbb{R}$ is convex iff for every $x < y < z$ in X,

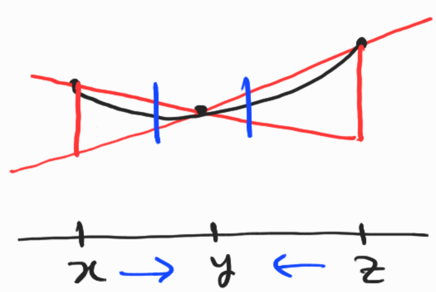
$$\frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(x)}{z-x} \le \frac{f(z)-f(y)}{z-y}.$$



The above problem is just algebra, do it yourself.

Problem

$f: X \to \mathbb{R}$, X open interval. Show that f is continuous on X.



For any two distinct points x, y in X, define
$$s(x,y) = \frac{f(y)-f(x)}{y-x}.$$

We can show (in the same spirit as

the last problem) that

$x_1 \le x_2$ and $y_1 \le y_2$ implies

$$s(x_1, y_1) \le s(x_2, y_2).$$

Why? Because

$$s(x_1, y_1) \le s(x_2, y_1) \le s(x_2, y_2).$$

follows from last problem.

Now, pick any x, y in X, say $x < y$.

$$s(x_1, x_2) \le s(x, y) \le s(y_1, y_2)$$

$$\Rightarrow c_1 \le \frac{f(y) - f(x)}{y - x} \le c_2 \quad \text{---} (*)$$

$$\Rightarrow \underline{c_1 (y - x) \le f(y) - f(x) \le c_2 (y - x).}$$

Letting $y \to x^+$, Sandwich gives

$$\lim_{y \to x^+} f(y) = f(x).$$

Similarly, we can set $(*)$ for $y < x$

also, and the same argument leads to

$$\lim_{y \to x^-} f(y) = f(x).$$

<u>Problem</u> $f: \mathbb{R} \to \mathbb{R}$ convex, $f(0) \leq 0$.

Show that for $a, b > 0$,

$$f(a+b) \geq f(a) + f(b).$$

We have

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Putting $y = 0$, we get

$$\begin{aligned} f(\lambda x) &\leq \lambda f(x) + (1-\lambda) f(0) \\ &\leq \lambda f(x). \end{aligned}$$

$x = a+b$, $\lambda = \frac{a}{a+b} \in (0, 1)$ gives

$$\frac{a}{a+b} f(a+b) \geq f(a). \quad \text{———— (I)}$$

$x = a+b$, $\lambda = \frac{b}{a+b} \in (0, 1)$ gives

$$\frac{b}{a+b} f(a+b) \geq f(b). \quad \text{———— (II)}$$

(I) + (II) gives the desired inequality.

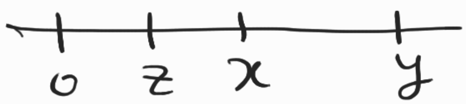
Problem

$f : \mathbb{R}^+ \to \mathbb{R}$ convex. $\lim_{x \to 0^+} f(x) = 0$.

Prove that $g(x) = \dfrac{f(x)}{x}$ is increasing.

Take any $0 < x < y$. We'll show that

$$\frac{f(x)}{x} \leq \frac{f(y)}{y}.$$



Take $0 < z < x$.

$$x = \lambda z + (1-\lambda) y$$

where $\lambda = \dfrac{y-x}{y-z}$.

Since f is convex,

$$f(x) \leq \lambda f(z) + (1-\lambda) f(y)$$

$$= \frac{y-x}{y-z} f(z) + \frac{x-z}{y-z} f(y).$$

Letting $z \to 0^+$, and using $\lim_{z \to 0^+} f(z) = 0$, we get

$$f(x) \leq \frac{y-x}{y} \times 0 + \frac{x}{y} f(y)$$

$$\Rightarrow \frac{f(x)}{x} \leq \frac{f(y)}{y}, \text{ as required.}$$

Jensen's ineq.

If f is convex on X, then for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0,1]$ such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1,$$

and any $x_1, x_2, \ldots, x_n \in X$, we have the following ineq:

$$f\left(\sum_{k=1}^n \lambda_k x_k\right) \leq \sum_{k=1}^n \lambda_k f(x_k).$$

For concave f, the reverse inequality holds.

Proof: By induction on n.

[If $\lambda_{n+1} = 1$, we are done. Else,

$\lambda'_k := \lambda_k/(1-\lambda_{n+1})$. $\lambda'_1 + \ldots + \lambda'_n = 1$.

Then $f\left(\sum_{k=1}^n \lambda_k x_k + \lambda_{n+1} x_{n+1}\right)$

$$\leq \lambda_{n+1} f(x_{n+1}) + (1-\lambda_{n+1}) f\left(\sum_{k=1}^n \lambda'_k x_k\right)$$

$$\leq \lambda_{n+1} f(x_{n+1}) + \sum_{k=1}^n (1-\lambda_{n+1}) \lambda'_k f(x_k).$$]

Applications of Jensen's ineq.

① A, B, C angles of a triangle.

Show that

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}.$$

$f(x) = \sin x \qquad f''(x) = -\sin x \le 0$

$\therefore$ f is concave.

Hence, by Jensen's inequality,

$$f\left(\frac{A+B+C}{3}\right) \geq \frac{f(A)+f(B)+f(C)}{3}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} \geq \frac{\sin A + \sin B + \sin C}{3}. \qquad \square$$

Strictly convex if strict ineq holds in the defn.

This is convex, but not strictly convex.

$f'' > 0 \Rightarrow f$ strictly convex

$\not\Leftarrow$ counterexample: x^4 on $\mathbb{R}$

f strictly convex, $f'' \geq 0$ but not > 0.

② AM-GM ineq.

For any $a_1, \dots, a_n > 0$,

$$\frac{1}{n}\sum_{k=1}^n a_k \geq \left(\prod_{k=1}^n a_k\right)^{1/n}.$$

This is equivalent to

$$\log\left(\frac{1}{n}\sum_{k=1}^n a_k\right) \geq \frac{1}{n}\sum_{k=1}^n \log a_k.$$

The above ineq. is just Jensen's inequality for $f(x) = \log x$, $x > 0$, which is a concave function.

Can you also prove

(i) weighted AM-GM ineq.?

(ii) Power mean inequality?

$$M_r = \left(\frac{1}{n}\sum_{k=1}^n a_i^r\right)^{1/r}, \quad r \in \mathbb{R} \setminus \{0\}.$$

$M_0 = GM$. $M_\infty = \max$, $M_{-\infty} = \min$.

Power mean ineq.: $r \leq s \Rightarrow M_r \leq M_s$.

③ $A, B, C \in \left(0, \frac{\pi}{2}\right)$, $A+B+C=\pi$.

Prove that

$$\cos A \cos B \cos C \le \frac{1}{8}.$$

$f(x) = \log \cos x$, $x \in \left(0, \frac{\pi}{2}\right)$

$f'(x) = -\tan x$, decreasing on $\left(0, \frac{\pi}{2}\right)$

$\therefore$ f is concave.

By Jensen's inequality,

$$\frac{1}{3} \sum_{cyc} f(A) \le f\left(\frac{A+B+C}{3}\right)$$

$$= \log \cos \frac{\pi}{3}$$

$$\Rightarrow \sum_{cyc} \log \cos A \le \log \frac{1}{8}$$

$$\Rightarrow \prod_{cyc} \cos A \le \frac{1}{8}. \quad \text{(Proved)}$$