Definitions of Continuity

Aditya Ghosh

July 2019

Warm Up

1. Suppose that $x_n \geq 0$ for all $n \geq 1$ and x_n converges to x. Show that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.
2. Suppose that x_n converges to x. Show that $\sin x_n$ converges to $\sin x$ and $\cos x_n$ converges to $\cos x$. Is it necessary that $\tan x_n$ converges to $\tan x$?

Solution:

1. Since $x_n \geq 0$ for all $n \geq 1$, so we must have $x \geq 0$. If $x = 0$, the result is easy to show ($x_n < \varepsilon^2 \implies \sqrt{x_n} < \varepsilon$). Next, assume that $x > 0$. Fix any $\varepsilon > 0$. Observe that,

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}. \tag{$*$}$$

   Since $x_n \to x$, we have an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon\sqrt{x}$ holds for all $n \geq N$. Hence, $(*)$ implies that $|\sqrt{x_n} - \sqrt{x}| < \varepsilon$ holds for every $n \geq N$. □

2. We shall use the fact that $|\sin t| \leq |t|$ which holds for all $t \in \mathbb{R}$. (It can be proved using a diagram of a unit circle.) Using this, we obtain

$$|\sin x_n - \sin x| = \left|2\sin\frac{x_n - x}{2}\cos\frac{x_n + x}{2}\right| \leq \left|2\sin\frac{x_n - x}{2}\right| \leq |x_n - x|.$$

   Since $|x_n - x| < \varepsilon$ holds for all but finitely many n, we get the same for $|\sin x_n - \sin x|$. Hence we conclude that $\sin x_n$ converges to $\sin x$. For cos, we can do as follows: since $y_n = \frac{\pi}{2} - x_n$ converges to $y = \frac{\pi}{2} - x$, we can say that $\sin y_n$ converges to $\sin y$. This is same as saying that $\cos x_n$ converges to $\cos x$. Next, for tan, observe that if $x \neq (2k+1)\frac{\pi}{2}$ for $k \in \mathbb{Z}$ then we get $\tan x_n = \dfrac{\sin x_n}{\cos x_n} \to \dfrac{\sin x}{\cos x} = \tan x$ as $n \to \infty$. But if $x = (2k+1)\frac{\pi}{2}$ for some $k \in \mathbb{Z}$ then it might happen that $\tan x_n$ does not converge at all. Here is a counter-example: $x_n = \frac{\pi}{2} - \frac{1}{n}$ if n is odd and $\frac{\pi}{2} + \frac{1}{n}$ if n is even. Convince yourself that although x_n converges to $\frac{\pi}{2}$, $\tan x_n$ does not converge at all.

Suppose x_n is a sequence that converges to x (in symbols, $x_n \to x$). We have seen earlier that $ax_n + b \to ax + b$, $|x_n| \to |x|$, $x_n^k \to x^k$ ($k \in \mathbb{N}$), $\sin x_n \to \sin x$, $\cos x_n \to \cos x$, $\sqrt{x_n} \to \sqrt{x}$ (provided $x_n \geq 0$), $\tan x_n \to \tan x$ (provided $\cos x \neq 0$). You can also show that for any polynomial $P(\cdot)$, we have $P(x_n) \to P(x)$. This idea generalizes to the notion of continuity:

> *Definition.* We say that $f(x)$ is continuous at $x = a$ if for every sequence x_n (belonging to the domain of f) that converges to a, it holds that $f(x_n)$ converges to $f(a)$.

As per our discussion above, it follows that $ax + b, |x|, x^k$ $(k \in \mathbb{N}), \sin x, \cos x$ are continuous at every $a \in \mathbb{R}$, $\sqrt{x}$ is continuous at every $a \geq 0$ and $\tan x$ is continuous at every $a \notin \{(2k+1)\frac{\pi}{2} : k \in \mathbb{Z}\}$. Next, we shall see a few properties of continuous functions.

> **Theorem.** Suppose that $f(x)$ and $g(x)$ are continuous at $x = a$. Then, the following functions are also continuous at $x = a$: $f(x) + g(x), f(x) - g(x)$, $cf(x)$ (where c is a constant), $f(x)g(x), |f(x)|, \max\{f(x), g(x)\}, \min\{f(x), g(x)\}$.

Proof. Take any sequence x_n that converges to a. Since f, g are continuous at $x = a$, we have $\lim_{n\to\infty} f(x_n) = f(a)$, $\lim_{n\to\infty} g(x_n) = g(a)$. Hence we get $\lim_{n\to\infty} f(x_n) + g(x_n) = f(a) + g(a)$. This shows that $f + g$ is continuous at $x = a$. In a similar manner we can show that $f - g, cf, fg, |f|$ are continuous at $x = a$. Next, let us consider $h(x) = \max\{f(x), g(x)\}$. Observe that for any $y, z \in \mathbb{R}$, we have

$$\max\{y, z\} = \frac{y + z + |y - z|}{2}.$$

Hence,

$$h(x) = \frac{1}{2}\Big(f(x) + g(x) + |f(x) - g(x)|\Big).$$

Complete the proof yourself. Do the same for $\min\{f(x), g(x)\}$. $\square$

Note that the last theorem allows us to construct a large class of continuous functions from the elementary ones, e.g. any polynomial, $|\sin x|, \sqrt{1 - x^2}, \dfrac{x + \cos x}{x^2 + 1}$, etc.

Next, we shall see another definition of continuity. Suppose $f(x)$ is continuous at $x = a$. When we draw the graph of $f(x)$, what do you expect (for the function to be continuous at $x = a$)? It is expected that as x gets closer and closer to a, $f(x)$ gets closer and closer to $f(a)$. This intuition is formalised in the following definition:

> *Definition.* We say that $f(x)$ is continuous at $x = a$ if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in (a - \delta, a + \delta)$ it holds that $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.

In words, it says that $f(x)$ will be sufficiently close to $f(a)$ if we take x to be sufficiently close to a. To have $f(x)$ to be at a distance less than ε from $f(a)$, we need x to be at a distance less than δ from a. The definition says that for every $\varepsilon > 0$, there exists such a $\delta > 0$.

One important point to note here is the order of the quantifiers 'for all' and 'there exists'. Why does their ordering matter? You should get an answer by observing the following statements:

(i) $\forall$ city C in India, $\exists$ a 6-digit number n such that n is the pincode of city C.

(ii) $\exists$ a 6-digit number n such that $\forall$ city C in India, n is the pincode of city C.

Can you give a real-life example of continuous function? There are plently of examples, e.g. your age as a function of time, your height as a function of your age (not rounded off to 'years'), speed of a car. Can you give a real-life example of a function which is not continuous? One example is the fare of any public transport (bus, metro) as a function of the distance from the place you boarded. (From one stoppage to another it jumps in discrete steps, it does not change continuously.)

One question that might come to your mind: how can we have two seemingly different definitions for the same thing (continuity)? We shall address this question shortly; before that let us do a few problems using each of the two approaches: sequential definition and $\varepsilon - \delta$ definition.

Problem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$|f(x) - f(y)| \leq \lambda |x - y| \text{ for all } x, y \in \mathbb{R}.$$

(Here $\lambda > 0$ is fixed.) Show that $f(x)$ is continuous (at $x = a$ for every $a \in \mathbb{R}$).

Proof. Fix any $a \in \mathbb{R}$, we shall show that $f(x)$ is continuous at $x = a$.

(i) Using sequential definition: Take any sequence x_n that converges to a. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon/\lambda$ holds for all $n \geq N$. Then for every $n \geq N$, we get $|f(x_n) - f(a)| \leq \lambda |x_n - x| < \varepsilon$. □

(ii) Using $\varepsilon - \delta$ definition: Fix any $\varepsilon > 0$. We choose $\delta = \varepsilon/\lambda$. Then observe that for every x such that $|x - a| < \delta$, we get $|f(x) - f(a)| \leq \lambda |x - a| < \lambda\delta = \varepsilon$. (Note, the choice of δ comes from a rough-work, which is just the above chain of inequalities. We had $|f(x) - f(a)| < \lambda\delta$, so we just set that to be equal to ε.) □

Problem. Suppose that $f : A \to B$ and $g : B \to C$ (so that $(g \circ f)(x) = g(f(x))$ is well-defined). If $f(x)$ is continuous at $x = a$ and $g(y)$ is continuous at $y = f(a)$, show that $h(x) = g(f(x))$ must be continuous at $x = a$.

Proof. (i) Using sequential definition: Take any sequence x_n that converges to a. Since $f(x)$ is continuous at $x = a$, we deduce that $f(x_n)$ converges to $f(a)$. Now, since $y_n = f(x_n)$ is a sequence that converges to $f(a)$, the continuity of $g(y)$ at $y = f(a)$ yields that $g(y_n)$ must converge to $g(f(a))$. This is same as saying that $g(f(x_n)) = h(x_n)$ converges to $g(f(a)) = h(a)$. Since this holds for every sequence x_n that converges to a, we conclude that $h(x)$ is continuous at $x = a$. $\square$

(ii) Using $\varepsilon - \delta$ definition: Fix any $\varepsilon > 0$. Since $g(y)$ is continuous at $y = f(a)$, there exists a $\delta > 0$ such that for every $y \in (f(a) - \delta, f(a) + \delta)$ we have $|g(y) - g(f(a))| < \varepsilon$, i.e.,

$$|y - f(a)| < \delta \implies |g(y) - g(f(a))| < \varepsilon. \qquad (\star)$$

Again, since $f(x)$ is continuous at $x = a$, we have a $\delta' > 0$ such that for every $x \in (a-\delta', a+\delta')$ we have $|f(x) - f(a)| < \delta$. Combining this with $(\star)$ we obtain

$$|x - a| < \delta' \implies |f(x) - f(a)| < \delta \implies |g(f(x)) - g(f(a))| < \varepsilon.$$

(We used $(\star)$ for $y = f(x)$.) Thus, we have shown that for every $\varepsilon > 0$ there exists $\delta' > 0$ such that $|g(f(x)) - g(f(a))| < \varepsilon$ holds whenever $|x - a| < \delta'$. $\square$

Problem. Consider the function $f(x) = \lfloor x \rfloor$ = largest integer less than or equal to x. Show that $f(x)$ is continuous at $x = a$ if and only if a is not an integer.

Proof. The case when $a \notin \mathbb{Z}$, is left as an exercise for the reader. (Hint: show that for any $a \notin \mathbb{Z}$ there exists a $\delta > 0$ such that f is constant in $(a - \delta, a + \delta)$.) Here we shall address the case when $a \in \mathbb{Z}$. We need to show that $f(x)$ is *not* continuous at $x = a$.

(i) Using sequential definition: Negation of the sequential definition is easy: we need to provide a sequence x_n such that x_n converges to a but $f(x_n)$ does not converge to $f(a)$. Here is one such sequence: $x_n = a - \frac{1}{n}$. Note that $f(x_n) = a - 1$ for all $n \geq 1$, but $f(a) = a$. $\square$

(ii) Using $\varepsilon - \delta$ definition: Writing the negation of the $\varepsilon - \delta$ definition might be a hurdle for the reader. Let us do it slowly. First we write the definition for $f(x)$ to be continuous at $x = a$:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in (a - \delta, a + \delta), \text{ it holds that } |f(x) - f(a)| < \varepsilon.$$

To negate this definition, start from the left and change each $\forall$ to $\exists$ and vice-versa. The correct negation of the above will be: $f(x)$ is not continuous at $x = a$ if

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0, \exists x \in (a - \delta, a + \delta) \text{ for which } |f(x) - f(a)| \geq \varepsilon.$$

We set this 'bad' ε to be $\frac{1}{2}$. Then, for every $\delta > 0$, we have $x = a - \frac{\delta}{2} \in (a - \delta, a + \delta)$ for which $|f(x) - f(a)| = |(a - 1) - a| = 1 \geq \varepsilon$. $\square$

It is now the perfect time to address the equivalence of the two definitions of continuity.

Problem. Consider the following properties:
(1) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ holds whenever $|x - a| < \delta$.
(2) For every sequence x_n that converges to a it holds that $f(x_n)$ converges to $f(a)$.
Show that a function f has property (1) if and only if it has property (2).

Proof. First we shall show that (1) $\Rightarrow$ (2). Suppose a function has property (1). Take any sequence that converges to a. We need to show that $f(x_n)$ converges to $f(a)$. Fix any $\varepsilon > 0$. By property (1), there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ holds whenever $|x - a| < \delta$. Since $x_n \to a$, there exists $N \in \mathbb{N}$ such that $|x_n - a| < \delta$ holds for every $n \geq N$. Then, for every $n \geq N$ we obtain $|f(x_n) - f(a)| < \varepsilon$. This completes the proof.

Next we show that (2) $\Rightarrow$ (1). Suppose a function has property (2). It is actually hard to give a $\delta > 0$ for any fixed $\varepsilon > 0$ where δ has the desired property. So we shall use the method of contradiction. Let, if possible, f be a function with property (2), which does not satisfy (1). The negation of (1) reads: there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in (a - \delta, a + \delta)$ for which $|f(x) - f(a)| \geq \varepsilon$. So, for each $n \in \mathbb{N}$, we set $\delta = \frac{1}{n}$ to get a 'bad' x in $(a - \frac{1}{n}, a + \frac{1}{n})$, which we call x_n. Therefore, x_n is a sequence such that $x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$ and $|f(x_n) - f(a)| \geq \varepsilon$ for every $n \in \mathbb{N}$. Since $a - \frac{1}{n} < x_n < a + \frac{1}{n}$, Sandwich tells us that x_n converges to a. But $f(x_n)$ being at least ε away from $f(a)$, we see that $f(x_n)$ can not converge to $f(a)$. This contradicts the fact that f has property (2) and hence completes the proof. □

Exercises

1. Suppose that $f(x)$ is continuous at $x = a$. Determine, with proof, a necessary and sufficient condition for $1/f(x)$ to be continuous at $x = a$.

2. Suppose that $f(x)$ and $g(x)$ are not continuous at $x = a$. Is it possible that $f(x) + g(x)$ is continuous at $x = a$?

3. Suppose that $f(x)$ is continuous at $x = a$, but $g(x)$ is not continuous at $x = a$. Is it possible that $f(x) + g(x)$ is continuous at $x = a$?

4. Suppose that $f(x)$ and $g(x)$ are not continuous at $x = a$. Is it possible that $f(x)g(x)$ is continuous at $x = a$?

5. Suppose that $f(x)$ is continuous at $x = a$, but $g(x)$ is not continuous at $x = a$. Is it possible that $f(x)g(x)$ is continuous at $x = a$?

6. Suppose f and g are functions such that $g \circ f$ is well-defined. Determine whether it is possible for $g(f(x))$ to be continuous at $x = a$ if

(a) $f(x)$ is discontinuous at $x = a$ but $g(y)$ is continuous at $y = f(a)$.

(b) $f(x)$ is continuous at $x = a$ but $g(y)$ is discontinuous at $y = f(a)$.

(c) $f(x)$ is discontinuous at $x = a$ and $g(y)$ is discontinuous at $y = f(a)$.

7. Define $f(x) = \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$. Discuss the continuity of $f(x)$.

   Note: For this problem, sequential approach is more useful than the $\varepsilon - \delta$ approach.

8. Define $f(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$. Discuss the continuity of $f(x)$.

   Note: For this problem, the $\varepsilon - \delta$ approach is more useful than the sequential approach.

9. Let $A = [1, 2) \cup (2, 3]$. Define a function $f : A \to \mathbb{R}$ as follows: $f(x) = x + 1$ if $x \in [1, 2)$ and $3 - x$ if $x \in (2, 3]$. Can you draw the graph of $f(x)$ *without lifting the pen*? Is the function discontinuous at $x = 2$?

10. Let $A = [1, 2] \cup (3, 4]$. Define a function $f : A \to \mathbb{R}$ as follows: $f(x) = x + 1$ if $x \in [1, 2]$ and $3 - x$ if $x \in (3, 4]$. Can you draw the graph of $f(x)$ *without lifting the pen*? Determine whether the function is continuous on A or not.

11. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If $f(x)$ is continuous at $x = 0$, then show that $f(x)$ is continuous at $x = a$ for every $a \in \mathbb{R}$. (Hint: First show that $f(0) = 0$ or 1. What happens if $f(0) = 0$? If $f(0) \neq 0$, then show that $f(x) \neq 0$ for every $x \in \mathbb{R}$.)

Hints/Answers

1. If f is continuous at $x = a$ and $f(a) \neq 0$ then we can show that $1/f$ is also continuous at $x = a$. (Note that it is also a necessary condition.) To prove this, you can use the sequential definition.

2. Yes. Take $f(x) = [x]$ and $g(x) = -[x]$. Both f and g are discontinuous at any $a \in \mathbb{Z}$, but $f + g$ is continuous everywhere.

3. No. Because if $h = f + g$ and f are both continuous at $x = a$ then their difference $h - f$ must also be continuous at $x = a$.

4. Yes. Take $a = 0$ and define $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ and $g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

5. Yes. Take $a = 0$ and define $f(x) = 0$ for all x, and $g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

6. (a) Yes. Take any function f which is discontinuous at $x = a$ and set $g \equiv 0$ (i.e. $g(x) = 0$ for all x).

   (b) Yes. Take $g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ and $f(x) = \sqrt{2}$ for all x. You can show that g is not continuous anywhere. But, $g(f(x)) = 0$ for all x, hence $g \circ f$ is continuous everywhere.

   (c) Yes. Set $a = 0$. Take $f(x) = g(x) = 1/x$ for $x \neq 0$ and define $f(0) = g(0) = 0$. Observe that $g(f(x)) = x$ for all x.

7. When $a \neq 0$, its easy to see that $f(x)$ is continuous $x = a$. We claim that f is discontinuous at $x = 0$. To prove our claim, it sufficies to give two sequences x_n and y_n such that both of them converge to 0, but $f(x_n)$ and $f(y_n)$ do not converge to the same limit. One possible example is the followng:

$$\text{Take } x_n = \frac{1}{2n\pi + \pi/2} \text{ and } y_n = \frac{1}{2n\pi - \pi/2}, \text{ for all } n \geq 1.$$

8. When $a \neq 0$, its easy to see that $f(x)$ is continuous $x = a$. We claim that f is continuous at $x = 0$ as well. Observe that for every x, we have $|f(x) - f(0)| = |\sin x| \leq |x| = |x - 0|$. So, we can take $\delta = \varepsilon$ to meet the requirement in the $\varepsilon - \delta$ definition.

9. It should be clear that f is continuous at $x = a$ for every $a \in [1, 2) \cup (2, 3]$. Only what can confuse you is the question of continuity at $x = 2$. However, since f is not defined at $x = 2$, this question does not arise, because $f(2)$ is not defined!. (This is somewhat controversial. For f to be continuous at $x = a$, it is universally accepted that x must belong to the domain of f. But for discontinuity, some authors (e.g. Rudin) define f to be discontinuous at $x = a$ only when a belongs to the domain and f is not continuous at a. I also prefer this definition and hence say that f is neither continuous nor discontinuous at $x = 2$.)

   Note, though the function f is continuous on its entire domain, one cannot draw the graph without lifting the pen. (Who told you that the graph of a continuous function must be drawn without lifting the pen!).

10. The function is continuous on its domain A. As pointed out above, the question of f being continuous at $x = a$ does not arise if $a \notin A$.

11. Carry out the steps given in the hint. Also see the next exercise set.