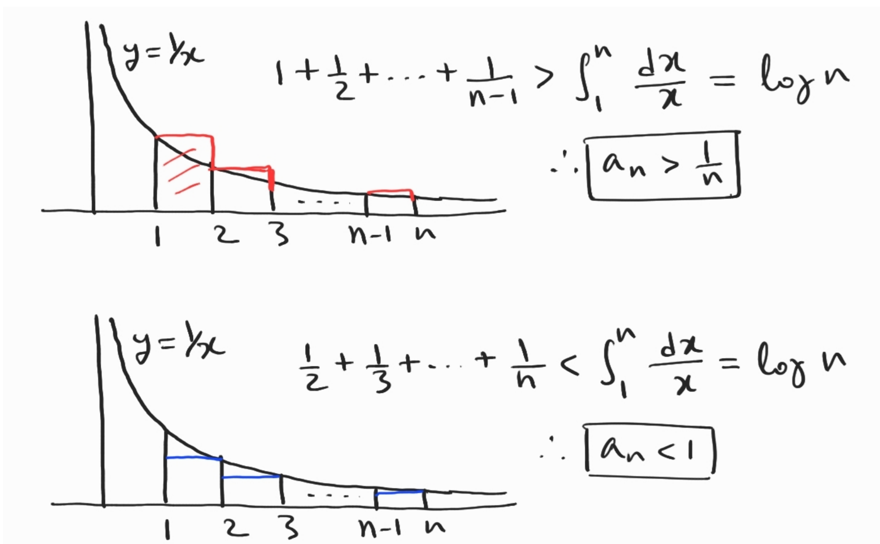
Integration : Theory and Problems (Day 4)

Aditya Ghosh

1. (a) For $n \geq 1$ define
$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n.$$
Prove that $\lim_{n\to\infty} a_n$ exists.

   (b) Hence (or otherwise) show that
$$\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = \log 2.$$



Solution. We have $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$. When $x \in [k, k+1]$, we have $1/k > 1/x > 1/(k+1)$, so
$$\frac{1}{k} = \int_k^{k+1} \frac{dx}{k} \geq \int_k^{k+1} \frac{dx}{x} \geq \int_k^{k+1} \frac{dx}{k+1} = \frac{1}{k+1}.$$
Hence,
$$\sum_{k=1}^{n-1} \frac{1}{k} \geq \sum_{k=1}^{n-1} \int_k^{k+1} \frac{dx}{x} = \int_1^n \frac{dx}{x} = \log n$$
which shows that $a_n > 1/n$ for each $n \geq 1$. On the other hand,
$$\sum_{k=1}^{n-1} \frac{1}{k+1} \leq \sum_{k=1}^{n-1} \int_k^{k+1} \frac{dx}{x} = \int_1^n \frac{dx}{x} = \log n$$
which gives $a_n < 1$ for all $n \geq 1$. Thus, a_n is bounded (between 0 and 1). But we could

not apply Sandwich to the bounds $1/n < a_n < 1$. However, we are lucky here: a_n is monotonically decreasing, as seen below.

$$a_{n+1} - a_n = \frac{1}{n+1} - (\log(n+1) - \log n) = \frac{1}{n+1} - \int_n^{n+1} \frac{dx}{x} = \int_n^{n+1} \Big(\underbrace{\frac{1}{n+1} - \frac{1}{x}}_{\leq 0}\Big) dx$$

which shows that $a_{n+1} \leq a_n$ for every $n \geq 1$. Since a_n is monotonically decreasing and bounded below, we can say that $\lim_{n\to\infty} a_n$ exists.

This limit is known as Euler-Mascheroni constant (γ). It does not have any other closed form expression, nor we know whether it is irrational or not (that is still an open problem).

Let us now do the second part of the problem. Consider the sequence

$$b_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}, \ n \geq 1.$$

Note that

$$\begin{aligned} b_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\ &= a_{2n} + \log(2n) - (a_n + \log n) = a_{2n} - a_n + \log 2. \end{aligned}$$

Since $\lim_{n\to\infty} a_n$ exists, say γ, then $\lim_{n\to\infty}(a_{2n} - a_n) = \gamma - \gamma = 0$. Hence we can say that

$$\lim_{n\to\infty} b_{2n} = \log 2.$$

For $b_{2n+1} - b_{2n} = 1/(2n+1) \to 0$, we can also see that

$$\lim_{n\to\infty} b_{2n+1} = \log 2.$$

This completes the argument. □

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, satisfying

$$f(x) = \frac{1}{t} \int_0^t (f(x+y) - f(y))\, dy$$

for all $x \in \mathbb{R}$ and all $t > 0$. Then show that $f(x) = cx$ for all x, where c is a constant.

Solution. We fix x write the given equation as

$$tf(x) = \int_0^t (f(x+y) - f(y))dy.$$

Since the integrand is continuous, we can apply FTC to differentiate both sides w.r.t. t and get

$$f(x) = f(x+t) - f(t).$$

Thus, we have $f(x+t) = f(x) + f(t)$ for every $x \in \mathbb{R}$ and $t > 0$. Putting $x = 0$, we get $f(0) = 0$. Then putting $x = -t$ where $t > 0$, we get $f(0) = f(-t) + f(t) \implies f(-t) = -f(t)$ for any $t > 0$. Now for $t < 0$, say $t = -s$ where $s > 0$, we can write

$$f(x+t) = f(x-s) = f(x) - f(s) = f(x) + f(-s) = f(x) + f(t).$$

Thus, we have now shown that $f(x+t) = f(x) + f(t)$ for every $x, t \in \mathbb{R}$. Now we can proceed in many ways. One way is to first show that $f(q) = qf(1)$ for every $q \in \mathbb{Q}$ and then use the continuity of f to conclude that $f(x) = cx$ for all $x \in \mathbb{R}$ where $c = f(1)$. Another way is to fix a $t > 0$ and rewrite the given equation as

$$tf(x) = \int_x^{x+t} f(y)dy - \int_0^t f(y)dy$$

and use FTC to say that the above RHS is differentiable w.r.t. x and hence so is the LHS. Therefore, f is differentiable, and the equation $f(x+y) = f(x) + f(y)$ gives, upon differentiating w.r.t. x, $f'(x+y) = f'(x)$ for any $y \in \mathbb{R}$. Thus, f' is a constant function on $\mathbb{R}$ and hence $f(x) = cx + d$ for all x where $f' \equiv c$. Putting $x = 0$, we get $d = 0$. □

3. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function such that for all $x \in (0, \infty)$, it holds that $f(2x) = f(x)$. Show that the function g defined by the equation

$$g(x) = \int_x^{2x} \frac{f(t)}{t} dt \quad \text{for } x > 0$$

is a constant function.

Solution. Fix $x_0 > 0$ and pick $0 < a < x_0$ and let $F(x) = \int_a^x f(t)/t \, dt$. Then for any $x > a$ we have $F'(x) = f(x)/x$ and since $g(x) = F(2x) - F(x)$, we get

$$g'(x) = 2F'(2x) - F'(x) = 2\frac{f(2x)}{2x} - \frac{f(x)}{x} = 0.$$

In particular, we get $g'(x_0) = 0$. Thus $g'(x_0) = 0$ for every $x_0 \in \mathbb{R}$ and hence g must be a constant function. □

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose that for all $x, y \in \mathbb{R}$, the function f satisfies $f'(x) - f'(y) \le 3|x - y|$. Show that for all $x, y \in \mathbb{R}$ we must have

$$|f(x) - f(y) - f'(y)(x - y)| \le 1.5(x - y)^2.$$

Also find the largest and smallest possible values of $f''(x)$.

Solution. First, let $x > y$. For any $t > y$, we have $f'(t) - f'(y) \le 3|t - y| = 3(t - y)$. Integrating both sides of this inequality w.r.t. t to arrive at

$$\int_y^x (f'(t) - f'(y))\, dt \le \int_y^x 3(t - y)dt$$

which gives

$$f(x) - f(y) - f'(y)(x - y) \le 3(x - y)^2/2. \tag{1}$$

Again, for any $t > y$, we also have $f'(y) - f'(t) \le 3|y - t| = 3(t - y)$. Integrating both sides of this inequality w.r.t. t to arrive at

$$\int_y^x (f'(y) - f'(t))\, dt \le \int_y^x 3(t - y)dt$$

which gives us

$$f(y) - f(x) - f'(y)(y - x) \le 3(x - y)^2/2. \tag{2}$$

Combining (1) and (2) we showed that for any $x > y$,

$$|f(x) - f(y) - f'(y)(y - x)| \le 1.5(x - y)^2.$$

For $x < y$, a similar argument applies (check!). And for $x = y$ it is trivial.

Again, note that we have here $|f'(x) - f'(y)| \le 3|x - y|$ for all $x, y \in \mathbb{R}$, from which we can easily show that $|f''(x)| \le 3$ for every $x \in \mathbb{R}$. Now we can show by means of an example that both of these extreme values (± 3) can be attained: for $f(x) = 3\cos x$ the given inequality holds, since $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$, and we have $f''(x) = -3\cos x$ whose maximum possible value is 3 and minimum possible value is -3. $\square$

5. Let $f : [0, 1] \to \mathbb{R}$ be differentiable. Suppose that $0 \le f'(x) \le 2f(x)$ holds for every $x \in [0, 1]$ and $f(0) = 0$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

Solution. Note that $(e^{-2x}f(x))' = e^{-2x}(f'(x) - 2f(x)) \le 0$ for every $x \in [0, 1]$. Hence $g(x) = e^{-2x}f(x)$ is a decreasing function. We find out that

$$0 \ge \int_0^x g'(t)dt = g(x) - g(0) = e^{-2x}f(x).$$

Thus, $f(x) \leq 0$ for all $x \in [0,1]$. But $f'(x) \geq 0$ implies that f is increasing and hence for any $x \in [0,1]$, $f(x) \geq f(0) = 0$. Hence $f \equiv 0$. $\square$

6. Let $f : [0,\infty) \to (0,\infty)$ be a continuously differentiable function. Prove that it is not possible that $f'(x) \geq (f(x))^2$ holds for all $x \geq 0$.

Solution. Using the given inequality we get, for any $x \geq 0$,

$$x = \int_0^x dt \leq \int_0^x \frac{f'(t)}{f(t)^2} dt = \frac{1}{f(0)} - \frac{1}{f(x)} < \frac{1}{f(0)}.$$

But x can be taken to be larger than $1/f(0)$, producing a contradiction. $\square$

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0, f'(0) = 3$ and $f''(x) = f(x)$ for all $x \in \mathbb{R}$. Find $f(\log 2019)$.

Solution. First we multiply both sides of the given equation with $f'(x)$ to get $f'(x)f''(x) = f(x)f'(x)$ and now we can integrate! Hence we get

$$\int_0^x f'(t)f''(t)dt = \int_0^x f(t)f'(t)dt \implies f'(x)^2 - f'(0)^2 = f(x)^2 - f(0)^2.$$

Using the given initial values, we get

$$f'(x)^2 = f(x)^2 + 3^2, \ x \in \mathbb{R}.$$

This says that $f'(x) = \pm\sqrt{f(x)^2 + 3^2}$. Is it possible that $f'(a) = -\sqrt{f(a)^2 + 3^2}$ for some $a \in \mathbb{R}$? If this happens, then from $f'(0) > 0$ and $f'(a) < 0$ we can say that there must exist $b \in \mathbb{R}$ such that $f'(b) = 0$ which is possible iff $f(b)^2 = -3^2$ which is not possible. Therefore, we conclude that

$$f'(x) = \sqrt{f(x)^2 + 3^2}, \ x \in \mathbb{R}.$$

Now we can integrate to find $f(x)$. We have,

$$x = \int_0^x dt = \int_0^x \frac{f'(t)}{\sqrt{f(t)^2 + 3^2}} dt = \log\left(f(t) + \sqrt{f(t)^2 + 3^2}\right)\Big|_{t=0}^{t=x}$$

which gives $f(x) + \sqrt{f(x)^2 + 3^2} = 3e^x$ for every $x \in \mathbb{R}$. Hence

$$\sqrt{f(x)^2 + 3^2} - f(x) = \frac{3^2}{\sqrt{f(x)^2 + 3^2} + f(x)} = 3e^{-x}$$

and hence $f(x) = \frac{3}{2}(e^x - e^{-x})$. Ans: $\frac{3}{2}\left(2019 - \frac{1}{2019}\right)$. $\square$

8. Let f be a twice differentiable function satisfying

$$f(x) + f''(x) = -xg(x)f'(x)$$

for every $x \in \mathbb{R}$, where $g(x) \geq 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is bounded.

Solution. First we write

$$f(x)f'(x) + f'(x)f''(x) = -xg(x)f'(x)^2.$$

For $x > 0$, the above RHS is ≤ 0, hence for any $x > 0$ we have

$$\int_0^x f(t)f'(t)dt + \int_0^x f'(t)f''(t)dt \leq 0$$

which gives $f(x)^2 \leq f(x)^2 + f'(x)^2 \leq f(0)^2 + f'(0)^2 = M^2$ say, where $M > 0$. Hence $|f(x)| \leq M$ for every $x > 0$.

For $x < 0$, we have $-xg(x)f'(x)^2 \geq 0$ and hence[1]

$$\int_x^0 f(t)f'(t)dt + \int_x^0 f'(t)f''(t)dt \geq 0$$

which gives $f(x)^2 \leq f(x)^2 + f'(x)^2 \leq f(0)^2 + f'(0)^2 = M^2$ say, where $M > 0$. Hence $|f(x)| \leq M$ for every $x < 0$. $\square$

Comment. You should have noted by now that the common idea in the last few problems is to multiply both sides of the given equations with a suitable function and then just integrate it. It is indeed a simple idea that will help you again and again.

9. Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and $\displaystyle\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0,1)$ we have,

$$\left| \int_0^\alpha f(x)dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

Solution. Note that we can write

$$\int_0^\alpha f(x)dx = \alpha \int_0^1 f(\alpha y)dy = \alpha \int_0^1 \left(f(\alpha y) - f(y)\right) dy.$$

(Finding out the above identity is undeniably tricky, but as we will see, it is the most crucial part of this solution.)

[1]for $\displaystyle\int_0^x$ the inequality will get reversed. Recall that $h \geq 0 \implies \displaystyle\int_a^b h(t)dt \geq 0$ provided $a < b$.

Now, for any fixed y, we can apply MVT to say that

$$\frac{f(y) - f(\alpha y)}{y - \alpha y} = f'(c_y),$$

for some $c_y \in (\alpha y, y)$. We are given that f' is continuous on $[0, 1]$ and hence bounded. Let $M = \max_{0 \le x \le 1} |f'(x)|$. Then, for any $y \in [0, 1]$, we get from the above equation that

$$|f(y) - f(\alpha y)| = |f'(c_y)||y - \alpha y| \le M(1 - \alpha)y \tag{3}$$

and hence,

$$\begin{aligned}
\left| \int_0^\alpha f(x)dx \right| &= \left| \alpha \int_0^1 (f(\alpha y) - f(y)) \; dy \right| \\
&\le \alpha \int_0^1 |f(\alpha y) - f(y)| \; dy && \text{(triangle inequality for integrals)} \\
&\le \alpha(1 - \alpha)M \int_0^1 y dy && \text{(by the upper bound (3))} \\
&\le \frac{1}{8} \max_{0 \le x \le 1} |f'(x)| && \text{(by AM-GM inequality, } \alpha(1 - \alpha) \le 1/4\text{)}
\end{aligned}$$

as desired. □

10. Let $f : [0, 1] \to [0, \infty)$ be a continuous function satisfying

$$(f(t))^2 \le 1 + 2 \int_0^t f(u)du \text{ for every } t \in [0, 1].$$

Show that $f(t) \le 1 + t$ must hold for all $0 \le t \le 1$.

Solution. Define a function $g(t) = 1+2 \int_0^t f(u)du, \; t \in [0, 1]$. We are given that $f(t)^2 \le g(t)$ and $f(t) \ge 0$, so we get $f(t) \le \sqrt{g(t)}$. Now, by FTC,

$$g'(t) = 2f(t) \le 2\sqrt{g(t)}.$$

We can now integrate:

$$\sqrt{g(x)} - \sqrt{g(0)} = \int_0^x \frac{g'(t)}{2\sqrt{g(t)}} dt \le \int_0^x dt = x$$

and $g(0) = 1$, so we get $f(x) \le \sqrt{g(x)} \le x + 1$, which completes the proof. □

Extra topics

The following are some extra topics that we will discuss in the next few classes.

1. Improper integrals
2. L'hôpital's rule
3. Taylor's theorem
4. Convexity

And following are some notes from my blog on specific topics that you can read completely by yourself:

1. Uniform Continuity
2. Several routes connecting to the number e
3. A simple proof that pi is irrational
4. Stirling's formula for n!