Integration : Theory and Problems (Day 1)

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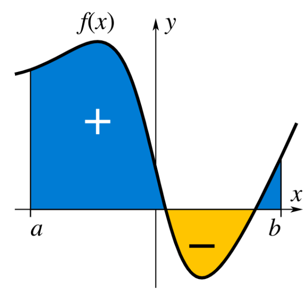
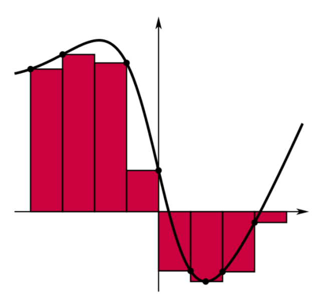
1 How to formally define the area?

You might have heard that $\int_a^b f(x)dx$ denotes the *area under the graph*[1] *of* $f(x)$ *from* $x = a$ *to* $x = b$. But what does that really mean? Recall how we learnt the concept of area since childhood. First we define a certain shape, suppose a square, to have a 'unit area'. Then for any rectangle, we measure the sides in that unit and say that area of the rectangle is length times breadth ($l \times b$), which essentially means that the area of the rectangle is lb times the unit area. For a triangle, we perform a similar procedure (we compare its area with a rectangle). But how to define area of some arbitrary shape? Intuition says that we should try to cover up the shape using those unit squares and find out *how many* of the unit squares are needed. Let us now try to make this intuition precise.

Given a function f defined on $[a, b]$, we wish to define the quantity $\int_a^b f(x)\,dx$ such that it represents the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$. The area above the x-axis adds to the total and that below the x-axis subtracts from the total.



For the time being, assume that the graph of $f(x)$ is 'simple', like the ones shown here. In order to approximate the area (as noted above), we divide the interval $[a, b]$ into some disjoint sub-intervals, suppose using the points $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and pick a point t_i in each $[x_{i-1}, x_i]$ such that $f(t_i)$ would be the height of a suitable rectangle that approximates the area under the curve within that sub-interval. Then we can use the sum of the areas of these rectangles to approximate the desired area.



However, there is a little problem. How do we choose those t_i's? Choosing t_i to be one of the endpoints might not always serve the purpose. Let us consider two extreme cases: when t_i is chosen such that $f(t_i)$ is the minimum or the maximum value of f within the sub-interval $[x_{i-1}, x_i]$. We know that if f is continuous on a closed bounded interval (like $[x_{i-1}, x_i]$) then it attains a minimum and a maximum within that interval. But not every bounded function f has this property[2], we should use sup and inf instead of max and min, respectively.

[1] Here graph of $f : [a, b] \to \mathbb{R}$ means a curve that consists of the points $\{(t, f(t)) : a \le t \le b\}$

[2] Take $f(x) = x(1 - x)$ if $x \neq 1/2$ and set $f(1/2) = 1/8$. Then f does not attain a maximum value within a sub-interval that contains the point $1/2$.

For each $1 \le i \le n$, we define

$$M_i = \sup\{f(t) : x_{i-1} \le t \le x_i\}, \text{ and } m_i = \inf\{f(t) : x_{i-1} \le t \le x_i\}.$$

Then, for any choice of t_i's, we have

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \le \sum_{i=1}^{n} M_i(x_i - x_{i-1}). \tag{1}$$

Given the partition $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$, the LHS of (1) is the worst under-estimate of the *desired area* with this partition (worst over all possible choices of t_i) and the RHS is the worst over-estimate. Keeping aside the t_i's for the moment, we might also write

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \text{desired area} \le \sum_{i=1}^{n} M_i(x_i - x_{i-1}). \tag{2}$$

Note that these under-estimate and over-estimate depend only on the partition P (and on f of course!), so we can denote them by $L(P, f)$ and $U(P, f)$, respectively. The quantity in the middle of (1) is called a *Riemann-sum approximation* of the desired area.

Example 1.1. Let us consider the function $f(x) = x$ on $[0, 1]$. Take the partition P that divides $[0, 1]$ into n intervals of equal length, i.e., $P_n = \{0, 1/n, 2/n, \ldots, 1\}$. Write $x_i = i/n$ for $0 \le i \le n$. Note that for $x \in [x_{i-1}, x_i]$, the maximum possible value of $f(x)$ is $M_i = f(x_i)$ and the minimum possible value is $m_i = f(x_{i-1})$. Hence,

$$U(P_n, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{n}\sum_{i=1}^{n} \frac{i}{n} = \frac{n(n+1)}{2n^2}$$

and

$$L(P_n, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n} f(x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n} \frac{i-1}{n} = \frac{n(n-1)}{2n^2}.$$

In this simple example, we already know what the area should be, because the region under the curve $y = f(x) = x$ for $x \in [0, 1]$ (and bounded below by the x-axis) is just a triangle which has area $1/2$. Observe that $U(P_n, f)$ is a slight over-estimate, while $L(P_n, f)$ is a slight under-estimate, which is exactly what we expect. In fact, letting $n \to \infty$, we see that both $U(P_n, f)$ and $L(P_n, f)$ converges to $1/2$.

Note that in the above example, we just considered a specific sequence of partitions. But there always is a plethora of partitions to choose from! Then how to develop a general notion of the area? Let us go back to equation (1) once again. The Riemann-sum approximation $\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ is always an estimate of the area, regardless of how we choose the t_i's. Intuition says that if the sub-intervals are made smaller and smaller, then this approximation will get closer and closer to the actual area. Having this in mind, let us try to perceive the following definition of integrals given by Riemann:

Definition 1.1 (Riemann's definition of integrability).
We say that f is (Riemann-)integrable on $[a,b]$ if there exists a real number A such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ has the property that the length of each sub-interval is less than δ, then it holds that

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right| < \varepsilon,$$

irrespective of the choice of the points t_i's where $t_i \in [x_{i-1}, x_i]$. If the above holds, we write

$$\int_a^b f(x)dx = A.$$

Informally, the above definition says that by making the length of each sub-interval sufficiently small, we can ensure the Riemann-sum approximation to be as close to $\int_a^b f(x)dx$ as we please. In particular, if we take n sub-intervals of equal length (i.e., each of length $\frac{b-a}{n}$) and let $n \to \infty$, we get the following theorem:

Theorem 1.1. *If f is integrable on $[a,b]$, then it holds that*

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right) = \int_a^b f(x)dx.$$

On one hand, this theorem can be used to calculate integrals of very simple functions (e.g., $x, x^2, e^x, \sin x$ etc.), while on the other hand, it can be used to calculate certain limits which can be expressed as the limit in the above LHS. This latter idea is extensively used for creating and solving competition problems. We shall see some interesting examples soon!

Corollary 1.1. *If f is integrable on $[0,1]$ then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx.$$

Question 1. *If $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \ell$, does it always imply that $\int_0^1 f(x)dx = \ell$?*

It is undeniable that even understanding the statement of Definition 1.1 takes a lot of effort, let alone the struggle of learning how to use it to prove that a given function is integrable. This is where Darboux's alternate definition (of Riemann integral) comes to our rescue. We consider all possible partitions of $[a,b]$ and try to find what are the *best under-estimate* and *best over-estimate* of the desired area, best over the choice of P, i.e., over the class of all partitions of $[a, b]$. Intuition suggests that best under-estimate is the largest one among all such under-estimates $L(P, f)$ and best over-estimate is the smallest one among all such over-estimates $U(P, f)$. Again, here a smallest or a largest one may not exist, so we use inf and sup :

$$\text{best over-estimate} = \inf U(P, f), \quad \text{best under-estimate} = \sup L(P, f) \tag{3}$$

where the infimum and the supremum are taken over P, i.e., over all possible partitions of $[a, b]$. Note that $\inf U(P, f)$ exists because for any partition P, the quantity $L(P, f)$ is a lower bound on the set of all possible values of $U(P, f)$. A similar argument shows why $\sup L(P, f)$ exists. Also note that the best over-estimate is always greater than or equal to the best under-estimate, but if the former is strictly bigger than the later, then how can we define the area? Having this notion in mind, we settle for the following definition of the area/integral:

Definition 1.2 (Darboux's definition of Riemann-integrability)**.**
We say that f is Riemann-integrable on $[a, b]$ if the 'best over-estimate' and the 'best under-estimate', as defined in (3) are equal and their common value is denoted by $\displaystyle\int_a^b f(x)dx$.

The reader who is not lost yet might wonder, how can there be two definitions of the same thing? Of course this is not the first time it is happening in this Calculus course, we already had seen two equivalent definitions of continuity ($\varepsilon - \delta$ definition and sequential definition). What we just need here is a proof that shows the above two definitions to be equivalent. There are such proofs, but let us skip that for now, since it will obviously be very much involved. If you are interested, you can look it up in any undergraduate-level textbook on Real Analysis.

It should however be noted that the above definition only attaches a *meaning* to the symbol $\int_a^b f(x)dx$, it does not give any method to calculate it. Moreover, the set of all partitions is so huge that even for very simple functions, it is notoriously difficult to verify the above definition, i.e., to show that $\inf U(P, f)$ and $\sup L(P, f)$ are equal. However, there is a result that is very handy when one tries to prove that a given function is integrable, which is as follows.

Result 1.1. *A function f is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$ holds.*

We shall not prove this result here either. However, let us use this result to find out some common classes of functions that are Riemann-integrable.

Result 1.2. *If f is monotone on $[a, b]$ then it is integrable on $[a, b]$ as well.*

Result 1.3. *If f is continuous on $[a, b]$ then it is integrable on $[a, b]$ as well.*

Proof of Result 1.2. Without loss of generality, we may assume that f is increasing. Then, for any partition $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$, we have

$$U(P, f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}), \text{ and } L(P, f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Hence, if we choose P such that the length of each sub-interval $[x_{i-1}, x_i]$ is small, say less than δ, then

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) \cdot \delta = \left(f(b) - f(a)\right) \cdot \delta. \end{aligned}$$

So by choosing $\delta > 0$ such that $\delta \cdot (f(b) - f(a)) \leq \varepsilon$, we are through. □

Proof of Result 1.3. Since f is continuous, it attains a maximum and a minimum value in each sub-interval. Now, we wish to make the sub-intervals very small such that in each of them the difference between the maximum and minimum value of f is small enough. To achieve this, uniform continuity would help.

Since f is continuous on this closed and bounded interval $[a, b]$, we know that f must be uniformly continuous on $[a, b]$. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in [a, b]$ such that $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Now, if we choose the partition P such that length of each sub-interval is less than this δ, then we know that $M_i - m_i$ is less than ε, for each i. Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon(b - a),$$

which completes the proof. (We could have started with $\varepsilon' = \varepsilon/(b - a)$ instead.) □

We shall assume the following results without proof. A curious reader can locate the proofs in any UG-level textbook on Real Analysis.

Result 1.4. *If f and g are integrable on $[a, b]$, then so are $f \pm g, cf$ (where c is a constant).*

Result 1.5. *If f is integrable on $[a, b]$, and g is continuous on the range of f, then $g \circ f$ is integrable.*

You are encouraged to use whatever learnt till now to answer the following questions.

Question 2. *If f is integrable on $[a, b]$, is it necessary that $|f|$ is also integrable?*

Question 3. *Suppose that f and g are integrable on $[a, b]$. Is it necessary that their product fg is also integrable on $[a, b]$? What about $\max\{f, g\}$ and $\min\{f, g\}$?*

Till now we have not seen any function which is not integrable. Following is a classical example of such kind.

Example 1.2 (Dirichlet function)**.** Let $f : [0, 1] \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Take any partition P. In each sub-interval, there is at least one rational and at least one irrational number, which implies that $M_i = 1$ and $m_i = 0$ holds for each i. Therefore,

$$U(P, f) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = 1 \cdot (1 - 0) = 1, \quad L(P, f) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = 0,$$

for any partition P. Hence we can say that f is not integrable on $[0, 1]$, by noting that f does not meet the requirements in Darboux's definition.

Remark 1.1. If we change the value of f at just one point, that does not have any influence on the integrability of f or on the value of the integral. Hence, if a function f is continuous everywhere except at just one point inside $[a, b]$, then f would be integrable, provided f is bounded. (It requires a proof though, which we skip for now.)

Remark 1.2. Note that from the very beginning we have imposed a condition that f must be bounded. How to define integrals of functions such as $f(x) = \log x$ on the interval $[0, 1]$ or $g(x) = \tan x$ on the interval $[0, \pi]$ (with $f(0)$ and $g(\pi/2)$ being defined something forcibly)? Integral of such functions are called improper integrals and will be discussed later. For now you can just keep in mind that they are defined using limits, e.g.,

$$\int_0^1 \log x \; dx \stackrel{\text{def}}{=} \lim_{a \to 0^+} \int_a^1 \log x \; dx.$$

Question 4. *If f is continuous on $[a, b]$ except only at 10^{10} many points, will f be necessarily integrable?*

Question 5. *Is it possible to have a function f which is integrable but discontinuous at infinitely many points?*

If you know the distinction between countably infinite and uncountably infinite, try to answer the following question.

Question 6. *If we change the value of f at countably many points, will it have any influence on the integrability of f?*

2 Some basic properties

Result 2.1. *If f, g are integrable on $[a,b]$ then* $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Proof. As we mentioned earlier, there is a result which says that $f+g$ is integrable if f, g are integrable. Hence we can use Theorem 1.1 to get

$$\begin{aligned}
\int_a^b (f+g) &= \lim_{n\to\infty} h_n \sum_{k=1}^{n} \left(f\left(a+kh_n\right) + g\left(a+kh_n\right)\right) \quad \left(\text{where } h_n = \frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} h_n \sum_{k=1}^{n} f\left(a+kh_n\right) + \lim_{n\to\infty} h_n \sum_{k=1}^{n} g\left(a+kh_n\right) \\
&= \int_a^b f + \int_a^b g
\end{aligned}$$

where in the last step we used Theorem 1.1 again. $\square$

In a similar manner we can prove the following results using Theorem 1.1 (and the reader is strongly encouraged to write their proofs, before proceeding further).

Result 2.2. *Let f, g be integrable on $[a,b]$. Then,* $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$, *for any constants α, β.*

Result 2.3. *Let f be integrable on $[a,b]$. Then,* $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx$.

Result 2.4. *If f is integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$ then* $\int_a^b f(x)dx \geq 0$.

However, the proof of the following result involves $U(P,f)$ and $L(P,f)$, so we skip its proof for now.

Result 2.5. *Suppose that f is integrable on $[a,c]$ and on $[c,b]$. Then f must be integrable on $[a,b]$ and* $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Answers to the Questions 1 through 7

1. *No. Take, for instance, the Dirichlet function (in Example 1.2).*

2. *Yes, by applying Result 1.5, because $g(x) = |x|$ is continuous everywhere.*

3. *Yes, by applying Result 1.5, because we can write* $fg = ((f+g)^2 - (f-g)^2)/4, \max\{f,g\} = (f+g+|f-g|)/2$ *and* $\min\{f,g\} = (f+g-|f-g|)/2$.

4. *Yes, by repeated application of Remark 1.1.*

5. *Yes, it is possible. Take* $f(x) = 1$ *if* $x = 1/n$ *for some* $n \in \mathbb{N}$, *and* $f(x) = 0$ *otherwise. You may use the* $U(P,f) - L(P,f) < \varepsilon$ *approach to show that* f *is integrable. It can also be shown that* $\int_0^1 f(x)dx = 0$.

6. *If* f *is integrable and* g *is obtained from* f *by changing* f *at countably many points, then* g *need not be integrable. E.g., take* g *to be the Dirichlet function and* $f(x) = 1$ *for all* x.

7. *No, because we may start with* $f \equiv 0$ *and change its value at just one point. For instance, take* $f(x) = 0$ *if* $x \neq 1$ *and* $f(1) = 2$. *Then,* $\int_0^2 f(x)dx = 0$, *but* f *is not identically zero.*

3 Some problems

Problem 3.1. *(MVT for integrals) Let* f *be continuous and* g *be integrable on* $[a,b]$ *and assume that* g *is positive. Show that there exists* $c \in [a,b]$ *such that*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Solution. Since f is continuous on $[a,b]$, we know that f attains a minimum and a maximum on $[a,b]$, say $f(m) \leq f(x) \leq f(d)$ for every $x \in [a,b]$. Since g is positive, we have

$$f(m)g(x) \leq f(x)g(x) \leq f(d)g(x), \text{ for every } x \in [a,b],$$

and hence

$$f(m)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq f(d)\int_a^b g(x)dx$$

Now the conclusion follows from the intermediate value property of f. □

Question 7. *Suppose that* f *is integrable and non-negative on* $[a,b]$. *If* $\displaystyle\int_a^b f(x)dx = 0$, *is it necessary that* f *must be identically zero on* $[a,b]$*?*

As you might have guessed, the answer to the above question is in the negative (try to find a counter-example then). However, if we impose an additional assumption that f must be continuous, then the following result holds.

Problem 3.2. *Let* f *be continuous and non-negative on* $[a,b]$. *If* $\displaystyle\int_a^b f(x)dx = 0$, *then show that* f *must be identically zero on* $[a,b]$.

Intuition says that if f is strictly positive at some point, then there will be a part of the curve $y = f(x)$ that lies strictly above the x-axis, which implies that the area under the curve can not be zero. Let us now try to write a rigorous proof, with the help of ε and δ.

Solution. Let, if possible, there be a point $c \in (a, b)$ such that $f(c) > 0$. By continuity, for $\varepsilon = f(c)/2$, there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for every $|x - c| < \delta$. Note that

$$|f(x) - f(c)| < \varepsilon \implies \varepsilon > |f(x) - f(c)| > |f(c)| - |f(x)| \implies f(x) > f(c)/2.$$

Since $f \geq 0$ on $[a, b]$, we have

$$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f \geq \int_{c-\delta}^{c+\delta} f(x)dx \geq \int_{c-\delta}^{c+\delta} \frac{f(c)}{2} dx = f(c) \cdot \delta > 0$$

which violates the given condition that $\displaystyle\int_a^b f = 0$. Therefore, for $\displaystyle\int_a^b f(x)dx$ to be 0, we need $f(x) = 0$ for all $x \in (a, b)$.

Finally, the continuity of f ensures that $f(a) = 0 = f(b)$. □

Problem 3.3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies*

$$\int_0^1 f(x)(1 - f(x))dx = \frac{1}{4}.$$

What can you say about f?

Solution. First note that

$$\int_0^1 f(x)(1 - f(x))dx = \frac{1}{4} \iff \int_0^1 (f(x) - 1/2)^2 dx = 0.$$

Since the function $g(x) = (f(x) - 1/2)^2$ is continuous and non-negative, this implies that $g(x) = 0$, i.e., $f(x) = 1/2$ for every $x \in [0, 1]$. However, having no information on f outside the interval $[0, 1]$, we are unable to conclude anything about f beyond $[0, 1]$. □

Problem 3.4 (Cauchy-Schwarz inequality)**.** *Let f, g be integrable on $[a, b]$. Prove that*

$$\left(\int_a^b f(x)^2 dx\right)\left(\int_a^b g(x)^2 dx\right) \geq \left(\int_a^b f(x)g(x)dx\right)^2$$

Furthermore, if f, g are continuous, then equality holds if and only if $f(x) = \lambda g(x)$ for some constant λ and for all $x \in [a, b]$.

One way to prove this is to use Theorem 1.1, which I encourage you to write down. This proof, however, fails to provide a justification for the equality case here. So we shall give another proof below, which essentially mimics the proof of C-S inequality for real numbers.

Solution. For $t \in \mathbb{R}$ define

$$h(t) = \int_a^b (f(x) - tg(x))^2 dx = At^2 - 2Bt + C,$$

where

$$A = \int_a^b g(x)^2 dx, \ B = \int_a^b f(x)g(x)dx, \ C = \int_a^b f(x)^2 dx.$$

Now, $h(t)$ is a quadratic in t, which is always non-negative, with leading coefficient $A > 0$ (the case $A = 0$ is trivial, in view of the Result 3.2). Hence[3] it follows that the discriminant $4B^2 - 4AC$ must be non-positive, i.e.,

$$B^2 \leq AC.$$

This is precisely the C-S inequality that we wanted to show. For equality to hold, we must have discriminant equal to zero, which says that the function $h(t)$ has a real root, say $t = \lambda$. After a little algebra, this is seen to be same as saying that

$$h(\lambda) = \int_a^b (f(x) - \lambda g(x))^2 dx = 0.$$

Since the function $(f(x) - \lambda g(x))^2$ is continuous and non-negative, Result 3.2 implies that $f(x) - \lambda g(x)$ must be identically zero on $[a, b]$. $\square$

[3] We can express $h(t) = At^2 - 2Bt + C$ as

$$h(t) = \frac{(At - B)^2}{A} - \frac{4B^2 - 4AC}{4A}.$$

So, the minimum value of $h(t)$ is the **negative** of its discriminant, divided by $4A$. That minimum value has to be non-negative here, and since $A > 0$, its numerator must be non-negative too. Hence the discriminant must be non-positive.

Exercise 1 on Integration

1. Define $f(x) = \int_0^1 |t-x|t\,dt$, for $x \in \mathbb{R}$. Sketch the graph of $f(x)$. What is the minimum value of $f(x)$?

2. For any positive integer n, let $C(n)$ denote the number of points which have integer coordinates and lie inside the circle $x^2+y^2=n^2$. Show that the limit

$$\lim_{n\to\infty} \frac{C(n)}{n^2}$$

exists and also evaluate this limit. Can you explain the result intuitively?

3. Let f, g be polynomials of degree n such that $\int_0^1 x^k f(x)dx = \int_0^1 x^k g(x)dx$ holds for each $k = 0, 1, \ldots, n$. Show that $f = g$.

4. Let f, g be continuous and positive functions defined on $[0,1]$ satisfying

$$\int_0^1 f(x)dx = \int_0^1 g(x)dx.$$

Define $y_n = \int_0^1 \frac{(f(x))^{n+1}}{(g(x))^n}dx$, for every integer $n \geq 0$. Show that $\{y_n\}_{n\geq 0}$ is an increasing sequence.

5. Suppose that f is integrable on $[a,b]$. Define

$$F(x) = \int_a^x f(t)dt, \text{ for } a \leq x \leq b.$$

Then, (i) F is continuous on $[a,b]$, and (ii) if f is continuous at $c \in [a,b]$, then F will be differentiable at c, with $F'(c) = f(c)$.

6. If f is differentiable on $[a,b]$ such that f' is continuous on $[a,b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

7. If f is continuous on $[a,b]$, show that $\int_a^b f(t)dt = f(c)(b-a)$ must hold for some $c \in (a,b)$.

8. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)dx = 1$. Show that there exists a point $c \in (0,1)$ such that $f(c) = 3c^2$.

9. Prove the inequalities: $\frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx \leq \frac{2\pi^2}{9}$.
