Integration : Theory and Problems (Day 3)

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1 Applications of the Fundamental Theorems of Calculus

1.1 Integrating by parts

Suppose that f, g are differentiable functions, such that their derivatives are integrable on $[a, b]$. Since $(fg)' = f'g + fg'$, we can write

$$\int_a^b f(x)g'(x)dx = \int_a^b \left((f(x)g(x))' - f'(x)g(x)\right)dx.$$

Now applying FTC, we get

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)g(x)dx.$$

This formula is commonly known as *integrating by parts*. It is particularly useful when it is difficult to find an anti-derivative of fg', but easy to do the same for $f'g$. For example, if $f(x) = \tan^{-1} x$ and $g(x) = x$, then note that it is difficult to directly find an anti-derivative of $f(x)g'(x) = \tan^{-1} x$, but for $f'(x)g(x) = \frac{x}{1+x^2}$ we immediately see that $\frac{1}{2}\log(1+x^2)$ is an anti-derivative. This is the key idea behind integration by parts: we split fg' into two parts, namely $(fg)'$ and $f'g$, each of which are easier to integrate.

An alternate form of integration by parts is found in Indian textbooks. Suppose that $g(x)$ is an anti-derivative of some function $h(x)$. Note that,

$$\int_a^b f(x)h(x)dx = \int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)g(x)dx.$$

Since g is an anti-derivative of h, we may write $g(x) = \displaystyle\int h(x)dx$. Then, the above equation takes the form

$$\int_a^b f(x)h(x)dx = \left(f(x)\int h(x)dx\right)\Bigg|_{x=a}^{x=b} - \int_a^b \left(f'(x)\int h(x)dx\right)dx. \tag{1}$$

Here I emphasise on the fact that $\int h(x)dx$ is just a short-hand notation for *an anti-derivative of h*; it does not represent any area or anything like that. This is why it will be wrong to write that

$$\int_a^b f(x)h(x)dx = \left(f(x)\int_a^b h(x)dx\right)\Bigg|_{x=a}^{x=b} - \int_a^b \left(f'(x)\int_a^b h(x)dx\right)dx,$$

even though the by parts formula for indefinite integration reads

$$\int f(x)h(x)dx = \left(f(x)\int h(x)dx\right) - \int \left(f'(x)\int h(x)dx\right) dx.$$

Remark 1.1. *While calculating the indefinite integral of h in (1), we need not put an arbitrary constant c, because it would get cancelled out (can you see why?).*

Example 1.1. *Calculate* $\int_0^1 \log x \; dx$.

Solution. Note that this is actually an improper integral, since log is unbounded near 0. So let us instead calculate $\int_a^1 \log x \; dx$ first, and then we will take a limit at $a \to 0^+$. To evaluate the last integral, we integrate by parts. Set $f(x) = \log x$ and $g(x) = x$ and write

$$\int_a^1 \log x \; dx = \int_a^1 fg' = f(x)g(x)\Big|_{x=a}^{x=1} - \int_a^1 f'g = x\log x\Big|_{x=a}^{x=1} - \int_a^1 x\frac{1}{x} \; dx = -a\log a + a - 1.$$

Now we can show that[1] $\lim_{a\to 0^+} a\log a = 0$. Hence $\int_0^1 \log x \; dx = -1$. □

1.2 Substitution formula

Suppose that f, g are differentiable functions such that $(f \circ g)(x) = f(g(x))$ is well-defined, say for $x \in [a, b]$. Then the chain rule of differentiation states that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Integrating this equation from $x = a$ to b, we obtain (using FTC)

$$\int_a^b f'(g(x))g'(x)dx = \int_a^b (f \circ g)'(x)dx = f(g(b)) - f(g(a)) = \int_{g(a)}^{g(b)} f'(u)du.$$

This is exactly the substitution formula that you are possibly aware of, from high-school textbooks. Usually in high-school, our teachers tell the same thing in a slightly different manner. They say the following:

> Substitute $u = g(x)$, so that $du = g'(x)dx$. Now $u = g(a)$ when $x = a$, and $u = g(b)$ when $x = b$. Hence
>
> $$\int_a^b f'(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f'(u)du$$

[1] e.g. by L'hôpitals rule, or using the fact that $e^y \geq y^2/2$ which implies $a\log a = -y/e^y \to 0$ as $y = -\log a \to \infty$.

While the above is sufficient for a first course in Calculus, it is now a good time to learn rigorously when the above holds, and when it does not. If you are still under the impression that the above holds *always*, let me weaken your confidence for a moment, by giving the following example.

Example 1.2. *Let $\phi(\theta) = \tan\theta$ and $f(x) = 1/(1+x^2)$. Observe that*

$$\int_0^\pi f(\phi(\theta))\phi'(\theta)d\theta = \int_0^\pi \frac{\sec^2\theta}{1+\tan^2\theta}d\theta = \int_0^\pi d\theta = \pi.$$

Now, if we blindly apply the substitution formula, we get

$$\int_0^\pi f(\phi(\theta))\phi'(\theta)d\theta = \int_{\phi(0)}^{\phi(\pi)} f(x)dx = \int_0^0 f(x)dx = 0.$$

But this is absurd, because it yields that $\pi = 0$. Where is the mistake?

You should try yourself finding out what's wrong in the above example, I shall answer it later (in this note though). By now you should be convinced that we need to check some conditions before applying the substitution formula. The following theorem answers that, by putting minimal conditions on the function for the substitution (here, $u = \phi(t)$).

Theorem 1.1. *Let $\phi : [a,b] \to I$ be a differentiable function, where $I \subseteq \mathbb{R}$ is an interval. Suppose that ϕ' is integrable on $[a,b]$. Let $f : I \to \mathbb{R}$ be a continuous function. Then,*

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx.$$

Proof. Choose any $c \in I$ and define

$$F(y) = \int_c^y f(x)dx, \ y \in I.$$

Since f is continuous, we can apply FTC-I (or the Leibniz rule) which tells us that $F(y)$ must be differentiable w.r.t. y on I, with $F'(y) = f(y)$. Hence, using the chain rule of differentiation,

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_a^b F'(\phi(t))\phi'(t)dt = \int_a^b (F\circ\phi)'(t)dt.$$

Applying FTC-II here, we can say that

$$\int_a^b (F\circ\phi)'(t)dt = F(\phi(b)) - F(\phi(a)) = \int_{\phi(a)}^{\phi(b)} f(x)dx.$$

The last equality follows from the definition of F. □

There is another formulation of the substitution formula, where we don't restrict f to be continuous (we just need f to be integrable); however we need pay it off by restricting ϕ to be bijective. This version is stated in the following theorem.

Theorem 1.2. *Let $\phi : [a, b] \to I$ be a differentiable, bijective function, where $I \subseteq \mathbb{R}$ is an interval. Suppose that ϕ' is integrable on $[a, b]$. Let $f : I \to \mathbb{R}$ be an integrable function. Then,*

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx.$$

Proof of the last theorem is not as simple as the first one, let us skip it for now. Let us now go through some more examples.

Example 1.3. *Suppose we want to calculate $\int_0^2 x\cos(x^2+1)\,dx$. Here we make the substitution $u = \phi(x) = x^2 + 1$. The substitution formula gives*

$$\int_0^2 t\cos(t^2+1)\,dt = \int_0^2 \frac{1}{2}\cos(\phi(t))\phi'(t)\,dt = \frac{1}{2}\int_{\phi(0)}^{\phi(2)} \cos x dx = \frac{\sin 5 - \sin 1}{2}.$$

Note that in this example we used the substitution formula from <ins>*left to right*</ins>*.*

Example 1.4. *Suppose we want to calculate $\int_0^1 \sqrt{1-x^2}\,dx$. As usual, we substitute $x = \sin\theta$ here. So, $f(x) = \sqrt{1-x^2}$ and $x = \sin\theta = \phi(\theta)$. Using the substitution formula, we get*

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_{\phi(0)}^{\phi(\pi/2)} f(x)\,dx = \int_0^1 f(\phi(t))\phi'(t)dt = \int_0^{\pi/2} \cos^2 u\,du = \frac{\pi}{4}.$$

The last integral is calculated using $2\cos^2 u = 1 + \cos 2u$. Note that in this example we used the substitution formula from <ins>*right to left*</ins>*. Note that this example actually gives us a proof of the fact that area of a circle is πr^2 where r is the radius of the circle.*

Using the substitution formula, we can prove the following simple results, which are frequently used to manipulate definite integrals.

1. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$
2. $\int_0^a f(x)dx = \int_0^a f(a-x)dx.$
3. $\int_0^{2a} f(x)dx = \int_0^a (f(x) + f(2a-x))dx.$
4. $\int_{-a}^a f(x)dx = \int_0^a (f(x) + f(-x))dx = \begin{cases} 0 & \text{if } f \text{ is an odd function} \\ 2\int_0^a f(x)dx & \text{if } f \text{ is an even function} \end{cases}.$

Proofs of these formulae are straightforward and hence left as an exercise for the reader.

Question 1. *Can you calculate the following integral to get the WiFi password?*

Example 1.2 (continuing from p. 3)**.** *One reason why the substitution formula failed in this example is that $\phi(\theta)=\tan\theta$ is not really a function from $[0,\pi]$ to an interval $I\subseteq\mathbb{R}$, because $\tan(\pi/2)$ is undefined. If instead you first break the integral into two parts, one from $(0,\pi/2)$ and another from $(\pi/2,\pi)$, then we have no conflict:*

$$\begin{aligned}
\pi=\int_0^{\pi}\frac{\sec^2\theta}{1+\tan^2\theta}d\theta &= \int_0^{\pi} f(\phi(\theta))\phi'(\theta)d\theta\\
&= \lim_{a\to(\pi/2)^-}\int_0^{a} f(\phi(\theta))\phi'(\theta)d\theta+\lim_{b\to(\pi/2)^+}\int_b^{\pi} f(\phi(\theta))\phi'(\theta)d\theta\\
&= \lim_{a\to(\pi/2)^-}\int_{\phi(0)}^{\phi(a)} f(u)du+\lim_{b\to(\pi/2)^+}\int_{\phi(b)}^{\phi(\pi)} f(u)du\\
&= \lim_{a\to(\pi/2)^-}\left[\tan^{-1}(u)\right]_{\phi(0)}^{\phi(a)}+\lim_{b\to(\pi/2)^+}\left[\tan^{-1}(u)\right]_{\phi(b)}^{\phi(\pi)}\\
&= \lim_{a\to(\pi/2)^-}\tan^{-1}(a)-\lim_{b\to(\pi/2)^+}\tan^{-1}(b)=\pi/2-(-\pi/2)=\pi.
\end{aligned}$$

If you are still wondering exactly where did the proof fail, take $F(x)=\tan^{-1}(x)$ so that $F'=f$, and note that following equality marked with red fails here.

$$\int_0^{\pi}F'(g(x))g'(x)dx=\int_0^{\pi}(F\circ g)'(x)dx = F(g(\pi))-F(g(0))=\int_{g(0)}^{g(\pi)}F'(u)du.$$

Why does that fail? FTC does not apply here since the integrand is not even continuous at $t=\pi/2$, as

$$(F\circ\phi)(t)=\tan^{-1}\tan(t)=\begin{cases} t & \text{if } 0\le t\le\pi/2\\ t-\pi & \text{if } \pi/2\le t\le\pi.\end{cases}$$

Our bag of tools is now almost complete. In the next few classes we shall discuss more problems and touch upon a few other areas (e.g., improper integrals). A list of useful theorems and results is given at the end of this note. Let us end today's class with a few problems.

2 Some example problems

Problem 2.1. *Determine, with proof, the value of* $\lim_{n\to\infty} \sqrt[n]{n!}/n$.

Solution. Observe that

$$\lim_{n\to\infty} \log\left(\sqrt[n]{n!}/n\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\frac{k}{n} = \int_0^1 \log x \; dx.$$

We calculated this integral in the Example 1 in today's class. Thus

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\frac{k}{n} = \int_0^1 \log x \; dx = -1.$$

Finally, since $x \mapsto e^x$ is continuous we conclude that the desired limit is e^{-1}. (Ans.)

Problem 2.2. *Let* $f(x)$ *be a continuous function, whose first and second derivatives are continuous on* $[0, 2\pi]$ *and* $f''(x) \geq 0$ *for all* $x \in [0, 2\pi]$*. Show that*

$$\int_0^{2\pi} f(x)\cos x \; dx \geq 0.$$

Solution. The key idea is to integrate by parts.

$$\begin{aligned}
\int_0^{2\pi} f(x)\cos x \; dx &= \left[f(x)\int \cos x \; dx\right]_0^{2\pi} - \int_0^{2\pi} \left(f'(x)\int \cos x \; dx\right) \; dx \\
&= -\int_0^{2\pi} f'(x)\sin x dx. \qquad (*) \\
\text{(Integrating by parts again)} &= \left[-f'(x)\int \sin x dx\right]_0^{2\pi} - \int_0^{2\pi} \left[-f''(x)\int \sin x dx\right] dx \\
&= f'(2\pi) - f'(0) - \int_0^{2\pi} f''(x)\cos x dx \\
&= \int_0^{2\pi} f''(x) dx - \int_0^{2\pi} f''(x)\cos x dx.
\end{aligned}$$

In the last step we could use Fundamental Theorem of Calculus since f'' is continuous (and hence integrable). Now observe that

$$\cos x < 1 \implies f''(x)\cos x \leq f''(x) \implies \int_0^{2\pi} f''(x)\cos x \; dx \leq \int_0^{2\pi} f''(x) \; dx.$$

This gives us the desired inequality. □

Why is the continuity of f'' crucial in the above solution? Because otherwise we cannot use

$$\int_a^b g'(x)dx = g(b) - g(a)$$

with $g = f'$. By blindly using the above formula one may end up into horribly wrong results, such as:

$$\int_{-1}^1 \frac{1}{x^2}dx = -2.$$

(This is wrong, since integral of a positive function cannot be negative! By the way, can you give a correct evaluation of this integral?)

It turns out that here one can also give an alternate solution that does not rely upon the continuity of f''. We can write from ($*$) that

$$\int_0^{2\pi} f(x)\cos x \, dx = -\int_0^{2\pi} f'(x)\sin x dx = -\int_\pi^{2\pi} f'(x)\sin x dx - \int_0^\pi f'(x)\sin x dx.$$

Now, substitute $x = y + \pi$ in the first integral above, to arrive at

$$\begin{aligned}\int_0^{2\pi} f(x)\cos x \, dx &= -\int_0^\pi f'(y+\pi)\sin(y+\pi) \, dy - \int_0^\pi f'(x)\sin x \, dx \\ &= \int_0^\pi f'(x+\pi)\sin x \, dx - \int_0^\pi f'(x)\sin x \, dx \\ &= \int_0^\pi \underbrace{(f'(x+\pi) - f'(x))}_{\geq 0 \text{ since } f'' \geq 0} \underbrace{\sin x}_{\text{also } \geq 0} dx \geq 0.\end{aligned}$$

This completes the proof. □

Problem 2.3. *Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous function satisfying $xf(y) + yf(x) \leq 1$ for every $x, y \in [0,1]$. Show that*

$$\int_0^1 f(x)dx \leq \pi/4. \tag{2}$$

Find a function satisfying the given condition for which equality is attained here.

Solution. Substituting $x = \sin\theta$ in the integral, and using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we obtain

$$\begin{aligned}\int_0^1 f(x)dx &= \int_0^{\frac{\pi}{2}} f(\sin\theta)\cos\theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} f\Big(\sin\Big(\frac{\pi}{2} - \theta\Big)\Big)\cos\Big(\frac{\pi}{2} - \theta\Big)d\theta = \int_0^{\frac{\pi}{2}} f(\cos\theta)\sin\theta \, d\theta.\end{aligned}$$

Adding up these two expressions for the same integral,

$$2\int_0^1 f(x)dx = \int_0^{\frac{\pi}{2}} \left(f(\sin\theta)\cos\theta + f(\cos\theta)\sin\theta\right) d\theta \leq \int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}$$

which gives the desired inequality (2). For equality to hold, it is sufficient to have

$$f(\sin\theta)\cos\theta + f(\cos\theta)\sin\theta = 1$$

which holds for $f(x) = \sqrt{1-x^2}$. How to check that this function does satisfy the given condition for all $x, y \in [0,1]$? Well, we can apply the Cauchy-Schwarz inequality to conclude that for any $0 \leq x, y \leq 1$,

$$xf(y) + yf(x) = x\sqrt{1-y^2} + y\sqrt{1-x^2} \leq \sqrt{(1-y^2+y^2)(x^2+1-x^2)} = 1.$$

This completes the proof that $f(x) = \sqrt{1-x^2}$ is indeed a function satisfying the given condition for which equality is attained in (2). □

Problem 2.4. *Suppose f is a differentiable function such that $f(f(x)) = x$ holds for all $x \in [0,1]$. Also, $f(0) = 1$. For any $n \in \mathbb{N}$, find the value of*

$$\int_0^1 (x - f(x))^{2n} dx.$$

Solution. Since f is one-one and continuous, it must be monotone. And $f(0) = 1$ implies $f(1) = f(f(0)) = 0$. Therefore, f is monotonically decreasing and range of f is $[0,1]$. So we can apply the substitution formula, with $\phi = f$ $\big($i.e. substituting $x = f(t)\big)$ and get

$$\begin{aligned} I = \int_0^1 (x - f(x))^{2n} dx &= -\int_{f(0)}^{f(1)} (x - f(x))^{2n} dx \\ &= -\int_0^1 \big(f(t) - f(f(t))\big)^{2n} f'(t) dt. \end{aligned}$$

Next, using $f(f(t)) = t$, we can write the last integral as $\int_0^1 \big(f(t) - t\big)^{2n} f'(t)dt$. Therefore,

$$I = \int_0^1 (x - f(x))^{2n} dx = -\int_0^1 \big(f(x) - x\big)^{2n} f'(x) dx.$$

Adding up these two alternate expressions for I, we get

$$2I = \int_0^1 \big(f(x) - x\big)^{2n} (1 - f'(x)) dx.$$

Since this integrand is just the derivative of $\dfrac{(f(x)-x)^{2n+1}}{2n+1}$, we apply FTC to obtain

$$2I = \frac{1}{2n+1}\,(f(x)-x)^{2n+1}\Big|_{x=0}^{x=1} = \frac{2}{2n+1}$$

which implies that $\displaystyle\int_0^1 (x-f(x))^{2n}dx = \frac{1}{2n+1}$. (Ans)

Problem 2.5. *For $n = 1, 2, 3, 4$, define $I_n = \displaystyle\int_0^{n\pi} \frac{\sin x}{1+x}\, dx$. Arrange I_1, I_2, I_3, I_4 in the increasing order.*

Solution. We can roughly sketch the graph of $y = \dfrac{\sin x}{1+x}$ and using it we can guess the ordering. (The actual graph is shown in Figure 1.)

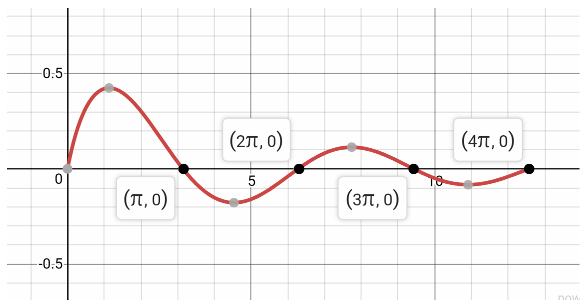

Figure 1: Graph of $y = \sin x/(1+x)$

Once we have guessed that the desired ordering is $I_1 > I_3 > I_4 > I_2$, it remains to show each of these inequalities one by one. First note that for any integer m,

$$\int_{m\pi}^{(m+1)\pi} \frac{\sin x}{1+x}dx = \int_0^{\pi} \frac{(-1)^m \sin y}{1+y+m\pi}dy.$$

(This can be seen by substituting $y = x - m\pi$.) Hence,

$$I_4 - I_2 = \int_{2\pi}^{3\pi} \frac{\sin x}{1+x}dx + \int_{3\pi}^{4\pi} \frac{\sin x}{1+x}dx = \int_0^{\pi}\left(\frac{\sin y}{1+2\pi+y} - \frac{\sin y}{1+3\pi+y}\right)dy > 0.$$

In the last step, we used the fact that integral of a positive and continuous function is positive. Similarly,

$$I_3 - I_1 = \int_{2\pi}^{3\pi} \frac{\sin x}{1+x}dx + \int_{\pi}^{2\pi} \frac{\sin x}{1+x}dx = \int_0^{\pi}\left(\frac{\sin y}{1+2\pi+y} - \frac{\sin y}{1+\pi+y}\right)dy < 0.$$

Finally,

$$I_3 - I_4 = -\int_{3\pi}^{4\pi} \frac{\sin x}{1+x}dx = \int_0^{\pi} \frac{\sin y}{1+3\pi+y}dx > 0.$$

Exercise 3 on Integration

1. Evaluate the following limits:

   (a) $\displaystyle\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}$ $(k \in \mathbb{N})$

   (b) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{(n+1)(n+2)\cdots(n+n)}$

   (c) $\displaystyle\lim_{n\to\infty} n^2 \left(\frac{1}{n^3+1^3} + \frac{1}{n^3+2^3} + \cdots + \frac{1}{2n^3} \right)$

   (d) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \log \binom{2n}{n}$

2. Evaluate the following integrals:

$$\int_{1/e}^{e} |\log x| \; dx, \quad \int_0^{\pi/2} \frac{1}{1+\tan^n x} \; dx, \quad \int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x} \; dx, \quad \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} \; dx.$$

3. Suppose f is continuous on $[0,1]$. Prove that

$$\int_0^{\pi} x f(\sin x) \; dx = \pi \int_0^{\pi/2} f(\sin x) \; dx.$$

   Hence (or otherwise) calculate

$$\int_0^{\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx.$$

4. Prove the following inequality

$$\int_0^{\pi} \left| \frac{\sin nx}{x} \right| dx \geq \frac{2}{\pi} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right).$$

5. For every positive integer n, evaluate the integrals

$$\int_0^{\pi/2} \sin^n x \; dx, \quad \int_0^{\pi/2} \cos^n x \; dx, \quad \int_0^{\pi/4} \tan^{2n} x \; dx, \text{ and } \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} \; dx.$$

6. For any $n \in \mathbb{N}$, evaluate the integral $\displaystyle\int_0^1 (1-x^2)^n dx$ and hence calculate the following sum

$$\frac{1}{1}\binom{n}{0} - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \cdots + (-1)^n \frac{1}{2n+1}\binom{n}{n}.$$

7. Let $f : [1,\infty) \to \mathbb{R}$ be defined by $f(x) = \displaystyle\int_1^x \frac{\log t}{1+t} dt$. Find all $x \in \mathbb{R}$ that satisfies the equation

$$f(x) + f(1/x) = 2.$$

8. Let f be continuous on $\mathbb{R}$. If $\int_{-a}^{a} f(x)dx = 0$ holds for every $a \in \mathbb{R}$, show that f must be an odd function.

9. Let $f : \mathbb{R} \to (0, \infty)$ be a continuously differentiable function which satisfies $f'(t) \geq \sqrt{f(t)}$ for all $t \in \mathbb{R}$. Show that for every $x \geq 1$,

$$\sqrt{f(x)} \geq \sqrt{f(1)} + \frac{1}{2}(x-1).$$

10. Let $f : [1, \infty) \to \mathbb{R}$ be a function satisfying $f(1) = 1$, and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

for every $x \geq 1$. Prove that $\lim_{x\to\infty} f(x)$ exists and this limit is less than $1 + \pi/4$.

11. Let $f(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. For every $n \in \mathbb{N}$ let f^n denote f composed n-times, i.e., $f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x)$. Evaluate $\int_0^1 f^{2020}(x)dx$.

12. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous. Define $a_n = \int_0^1 f(x+n)dx$, for every $n \geq 0$. Suppose also that $\lim_{n\to\infty} a_n = a$. Find the limit

$$\lim_{n\to\infty} \int_0^1 f(nx)dx.$$

13. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Prove that,

$$\lim_{n\to\infty} \int_a^b f(x) \sin nx dx = 0.$$
