# Exercise 3 on Integration

1. Evaluate the following limits:

(a) $\displaystyle\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}$ $(k \in \mathbb{N})$

(b) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(n+n)}$

(c) $\displaystyle\lim_{n\to\infty} n^2\left(\frac{1}{n^3+1^3} + \frac{1}{n^3+2^3} + \cdots + \frac{1}{2n^3}\right)$

(d) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\log\binom{2n}{n}$

2. Evaluate the following integrals:

$$\int_{1/e}^{e} |\log x|\ dx,\quad \int_0^{\pi/2} \frac{1}{1+\tan^n x}\ dx,\quad \int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x}\ dx,\quad \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\ dx.$$

3. Suppose $f$ is continuous on $[0,1]$. Prove that

$$\int_0^{\pi} xf(\sin x)\ dx = \pi\int_0^{\pi/2} f(\sin x)\ dx.$$

Hence (or otherwise) calculate

$$\int_0^{\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx.$$

4. Prove the following inequality

$$\int_0^{\pi} \left|\frac{\sin nx}{x}\right| dx \geq \frac{2}{\pi}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

5. For every positive integer $n$, evaluate the integrals

$$\int_0^{\pi/2} \sin^n x\ dx,\quad \int_0^{\pi/2} \cos^n x\ dx,\quad \int_0^{\pi/4} \tan^{2n} x\ dx,\ \text{ and }\quad \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\ dx.$$

6. For any $n \in \mathbb{N}$, evaluate the integral $\displaystyle\int_0^1 (1-x^2)^n dx$ and hence calculate the following sum

$$\frac{1}{1}\binom{n}{0} - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \cdots + (-1)^n\frac{1}{2n+1}\binom{n}{n}.$$

7. Let $f : [1,\infty) \to \mathbb{R}$ be defined by $f(x) = \displaystyle\int_1^x \frac{\log t}{1+t} dt$. Find all $x \in \mathbb{R}$ that satisfies the equation

$$f(x) + f(1/x) = 2.$$

8. Let $f$ be continuous on $\mathbb{R}$. If $\int_{-a}^{a} f(x)dx = 0$ holds for every $a \in \mathbb{R}$, show that $f$ must be an odd function.

9. Let $f : \mathbb{R} \to (0, \infty)$ be a continuously differentiable function which satisfies $f'(t) \geq \sqrt{f(t)}$ for all $t \in \mathbb{R}$. Show that for every $x \geq 1$,

$$\sqrt{f(x)} \geq \sqrt{f(1)} + \frac{1}{2}(x-1).$$

10. Let $f : [1, \infty) \to \mathbb{R}$ be a function satisfying $f(1) = 1$, and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

for every $x \geq 1$. Prove that $\lim_{x\to\infty} f(x)$ exists and this limit is less than $1 + \pi/4$.

11. Let $f(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. For every $n \in \mathbb{N}$ let $f^n$ denote $f$ composed $n$-times, i.e., $f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x)$. Evaluate $\int_0^1 f^{2020}(x)dx$.

12. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous. Define $a_n = \int_0^1 f(x+n)dx$, for every $n \geq 0$. Suppose also that $\lim_{n\to\infty} a_n = a$. Find the limit

$$\lim_{n\to\infty} \int_0^1 f(nx)dx.$$

13. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Prove that,

$$\lim_{n\to\infty} \int_a^b f(x) \sin nx dx = 0.$$

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## Solution to Exercise 3

1. Evaluate the following limits:

(a) $\lim_{n\to\infty} \dfrac{1^k + 2^k + \cdots + n^k}{n^{k+1}}$ $(k \in \mathbb{N})$

(b) $\lim_{n\to\infty} \dfrac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(n+n)}$

(c) $\lim_{n\to\infty} n^2\left(\dfrac{1}{n^3+1^3} + \dfrac{1}{n^3+2^3} + \cdots + \dfrac{1}{2n^3}\right)$

(d) $\lim_{n\to\infty} \dfrac{1}{n}\log\dbinom{2n}{n}$

*Solution.*

(a) It equals $\displaystyle\int_0^1 x^k dx = \frac{1}{k+1}$. (Ans)

(b) Taking log, we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\left(1+\frac{k}{n}\right) = \int_0^1 \log(1+x)dx = x\log x - x\Big|_{x=1}^{x=2} = 2\log 2 - 1.$$

So the desired limit equals $\exp(2\log 2 - 1) = 4/e$. (Ans)

(c) It equals $\displaystyle\int_0^1 \frac{1}{1+x^3}dx$. Evaluating this is usually carried out using a partial fraction decomposition: by assuming that

$$\frac{1}{(1+x)(1-x+x^2)} = \frac{A}{x+1} + \frac{Bx+C}{1-x+x^2}$$

is an identity we solve for $A, B, C$, and then use standard integrals. Another way is to do some algebra and cleverly write it as

$$\frac{1}{6}\int_0^1 \frac{1}{x+1}dx - \frac{1}{6}\int_0^1 \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int_0^1 \frac{1}{x^2-x+1}dx.$$

Anyway, these are some very standard methods that I hope you already are (or, going to be) familiar with them. The final answer is $\dfrac{1}{3}\log 2 + \dfrac{\pi}{3\sqrt{3}}$. (Ans)

(d) Since $\dbinom{2n}{n} = \displaystyle\prod_{k=1}^{n} \frac{n+k}{k}$, the given limit equals

$$\int_0^1 \log\left(1+\frac{1}{x}\right)dx = \int_1^2 \log x dx - \int_0^1 \log x dx = (2\log 2 - 1) - (-1) = \log 4. \quad \text{(Ans)}$$

2. Evaluate the following integrals:

$$\int_{1/e}^{e} |\log x|\ dx,\quad \int_0^{\pi/2} \frac{1}{1+\tan^n x}\ dx,\quad \int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x}\ dx,\quad \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\ dx.$$

*Solution.* The first one can be calculated as follows.

$$\begin{aligned}
\int_{1/e}^{e} |\log x|\ dx &= \int_{1/e}^{1} |\log x|\ dx + \int_{1}^{e} |\log x|\ dx \\
&= \int_{1/e}^{1} -\log x\ dx + \int_{1}^{e} \log x\ dx \\
&= x - x\log x\Big|_{x=1/e}^{x=1} + x\log x - x\Big|_{x=1}^{x=e} = 2(1-1/e). \qquad \text{(Ans)}
\end{aligned}$$

For the next one, the result $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ will help us, as follows.

$$I = \int_0^{\pi/2} \frac{1}{1+\tan^n x}\ dx = \int_0^{\pi/2} \frac{1}{1+\tan^n(\pi/2-x)}\ dx = \int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}\ dx.$$

Adding up these two expressions for $I$, we get $2I = \int_0^{\pi/2} 1dx = \pi/2 \implies I = \pi/4$. (Ans) To calculate the next one, we note that $(\sin x + \cos x)' = \cos x - \sin x$. So, writing $2\sin x = (\sin x + \cos x) - (\cos x - \sin x)$ does the trick:

$$\begin{aligned}
\int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x}\ dx &= \frac{1}{2}\int_0^{\pi/4} \frac{2\sin x}{\sin x + \cos x}\ dx \\
&= \frac{1}{2}\int_0^{\pi/4} 1\ dx - \frac{1}{2}\int_0^{\pi/4} \frac{(\sin x + \cos x)'}{\sin x + \cos x}\ dx \\
&= \frac{\pi}{8} - \frac{1}{2}\Big[\log(\sin x + \cos x)\Big]_{x=0}^{x=\pi/4} = \frac{\pi}{8} - \frac{1}{4}\log 2. \qquad \text{(Ans)}
\end{aligned}$$

Let us now calculate the last one.

$$I_1 = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\ dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\ dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\ dx$$

Adding up these two expressions for $I_1$ we get

$$2I_1 = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\ dx = \pi \int_{-1}^{1} \frac{1}{1+u^2} du = \pi\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$$

and hence $I_1 = \pi^2/4$. (Ans)

3. Suppose $f$ is continuous on $[0,1]$. Prove that

$$\int_0^{\pi} x f(\sin x)dx = \pi \int_0^{\pi/2} f(\sin x)dx.$$

Hence (or otherwise) calculate

$$\int_0^\pi \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx.$$

*Solution.* First we write

$$I = \int_0^\pi x f(\sin x) dx = \int_0^\pi (\pi - x) f(\sin(\pi - x)) dx = \int_0^\pi (\pi - x) f(\sin x) dx$$

and then adding up these two alternate expressions for the same integral, we get

$$2I = \pi \int_0^\pi f(\sin x) dx = 2\pi \int_0^{\pi/2} f(\sin x) dx$$

where in the last step we used $\int_0^{2a} f(x)dx = \int_0^a (f(x) + f(2a - x))dx$. □

Using the above formula/idea, we get

$$\int_0^\pi \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = \pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

Now using $\int_0^a f(x)dx = \int_0^a f(a - x)dx$,

$$I = \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = \int_0^{\pi/2} \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = \frac{1}{2} \int_0^{\pi/2} dx = \frac{\pi}{4}.$$

Therefore, the desired integral equals $\pi^2/4$. (Ans)

4. Prove the following inequality

$$\int_0^\pi \left| \frac{\sin nx}{x} \right| dx \geq \frac{2}{\pi} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right).$$

*Solution.* First we substitute $y = nx$ to write

$$\int_0^\pi \left|\frac{\sin nx}{x}\right| dx = \int_0^{n\pi} \left|\frac{\sin y}{y/n}\right| \frac{dy}{n} = \int_0^{n\pi} \left|\frac{\sin y}{y}\right| dy.$$

Now break the integral as the sum of integrals $\int_0^\pi$, $\int_\pi^{2\pi}$, etc. as follows.

$$\begin{aligned}\int_0^{n\pi}\left|\frac{\sin y}{y}\right|dy &= \sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\frac{|\sin y|}{y}dy \\ &\geq \sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\frac{|\sin y|}{k\pi}dy \quad (\text{since } (k-1)\pi < y < k\pi \implies 1/y > 1/k\pi) \\ &= \sum_{k=1}^{n}\frac{1}{k\pi}\int_0^{\pi}|\sin y|dy = \frac{2}{\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\end{aligned}$$

as required. □

**Corollary.** $\displaystyle\int_0^{\infty}\left|\frac{\sin y}{y}\right|dy = \lim_{T\to\infty}\int_0^{T}\left|\frac{\sin y}{y}\right|dy = \infty$. (Since $1+1/2+1/3+\cdots$ diverges.)

But, it is an interesting fact that $\displaystyle\int_0^{\infty}\frac{\sin y}{y}dy$ exists (which we will show in a later class) and, in fact, it equals $\pi/2$.

5. For every positive integer $n$, evaluate the integrals

$$\int_0^{\pi/2}\sin^n x\ dx,\ \int_0^{\pi/2}\cos^n x\ dx,\ \int_0^{\pi/4}\tan^{2n} x\ dx,\ \text{and}\ \int_0^{\pi/2}\frac{\sin(2n+1)x}{\sin x}\ dx.$$

*Solution.* Let me do the first two, and leave the rest for you. For $n \geq 1$, define

$$I_n = \int_0^{\pi/2}\sin^n x\ dx = \int_0^{\pi/2}\cos^n x\ dx.$$

For instance, $I_0 = \pi/2$, and $I_1 = 1$. How to calculate $I_n$ for a general $n$? The idea is to get a recursion for $I_n$ and then solve that recursion. For $n > 1$, we integrate by parts to get

$$\begin{aligned}I_n &= \int_0^{\pi/2}(\sin x)^{n-1}\cdot\sin x\ dx \\ &= \left[(\sin x)^{n-1}\int\sin x dx\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{d}{dx}(\sin x)^{n-1}\left(\int\sin x dx\right)dx \\ &= \left[-(\sin x)^{n-1}\cos x\right]_0^{\pi/2} + \int_0^{\pi/2}(n-1)(\sin x)^{n-2}\cos^2 x\ dx \\ &= 0 + \int_0^{\pi/2}(n-1)(\sin x)^{n-2}(1-\sin^2 x)\ dx = (n-1)(I_{n-2}-I_n).\end{aligned}$$

Thus, $I_n = (n-1)(I_{n-2}-I_n)$, which can also be written as

$$I_n = \frac{n-1}{n}I_{n-2},\ n \geq 2.$$

Now, for an even $n$, say $n = 2k$ where $k \geq 1$, we have

$$I_{2k} = \frac{2k-1}{2k} I_{2k-2} = \frac{2k-1}{2k}\frac{2k-3}{2k-2} I_{2k-4} = \cdots = \frac{1 \times 3 \times \cdots \times (2k-1)}{2 \times 4 \times \cdots \times 2k} I_0.$$

Similarly, for odd $n$, say $n = 2k+1$ where $k > 1$, we have

$$I_{2k+1} = \frac{2k}{2k+1} I_{2k-1} = \frac{2k}{2k+1}\frac{2k-2}{2k-1} I_{2k-3} = \cdots = \frac{2 \times 4 \times \cdots \times 2k}{3 \times 5 \times \cdots \times (2k+1)} I_1.$$

We can also write

$$I_n = \begin{cases} \dfrac{(2k-1)!!}{(2k)!!}\dfrac{\pi}{2} = \dbinom{2k}{k}\dfrac{\pi}{2^{2k+1}} & \text{if } n = 2k \geq 0, \\[2ex] \dfrac{(2k)!!}{(2k+1)!!} = \dfrac{2^{2k}}{2k+1}\dbinom{2k}{k}^{-1} & \text{if } n = 2k+1 \geq 1. \end{cases} \tag{3}$$

These integrals $(I_n)$ are commonly known as Wallis' integrals.

6. For any $n \in \mathbb{N}$, evaluate the integral $\int_0^1 (1-x^2)^n dx$ and hence calculate the following sum

$$\frac{1}{1}\binom{n}{0} - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \cdots + (-1)^n \frac{1}{2n+1}\binom{n}{n}.$$

*Solution.* Using the Binomial theorem,

$$(1-x^2)^n = \sum_{k=0}^{n} \binom{n}{k} (-x^2)^k.$$

Integrating both sides, and noting that the RHS being a finite summation we can pass the integral sign inside the summation, we get

$$\int_0^1 (1-x^2)^n dx = \sum_{k=0}^{n} \binom{n}{k} \int_0^1 (-x^2)^k dx = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1}.$$

Now, we can calculate the integral on the LHS directly (using by parts or by substitution) and hence get an expression for the sum on the RHS.

$$\int_0^1 (1-x^2)^n dx = \int_0^{\pi/2} (1-\sin^2\theta)^n \cos\theta \, d\theta = \int_0^{\pi/2} (\cos\theta)^{2n+1} \, d\theta = \frac{2 \times 4 \times \cdots \times 2k}{1 \times 3 \times \cdots \times (2n+1)}$$

where the last integral was evaluated using (3). Therefore,

$$\frac{1}{1}\binom{n}{0} - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \cdots + (-1)^n \frac{1}{2n+1}\binom{n}{n} = \frac{(2n)!!}{(2n+1)!!}. \tag{Ans}$$

7. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \displaystyle\int_1^x \frac{\log t}{1+t} dt$. Find all $x \in \mathbb{R}$ that satisfies the equation

$$f(x) + f(1/x) = 2.$$

*Solution.* For any $x > 1$, we calculate the following integral by substituting $u = 1/t$

$$\int_1^{1/x} \frac{\log t}{1+t} dt = \int_1^x \frac{\log(1/u)}{1+1/u} \frac{-1}{u^2} du = \int_1^x \frac{\log u}{1+u} \frac{du}{u}.$$

Therefore,

$$f(x) + f(1/x) = \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{\log t}{1+t} \frac{1}{t} dt = \int_1^x \frac{\log t}{t} dt = \frac{1}{2}(\log t)^2 \Big|_1^x = \frac{1}{2}(\log x)^2.$$

So, $f(x) + f(1/x) = 2 \iff (\log x)^2 = 4 \iff \log x = \pm 2 \iff x = e^2$ or $e^{-2}$. (Ans)

8. Let $f$ be continuous on $\mathbb{R}$. If $\displaystyle\int_{-a}^a f(x) dx = 0$ holds for every $a \in \mathbb{R}$, show that $f$ must be an odd function.

*Solution.* Using the formula $\displaystyle\int_{-a}^a f(x) dx = \int_0^a (f(x) + f(-x)) dx$, we get

$$\int_0^a g(x) dx = 0$$

for all $a \in \mathbb{R}$ where $g(x) = f(x) + f(-x)$. In a previous exercise we saw that this implies $g \equiv 0$, which here forces $f$ to be an odd function. □

9. Let $f : \mathbb{R} \to (0, \infty)$ be a continuously differentiable function which satisfies $f'(t) \geq \sqrt{f(t)}$ for all $t \in \mathbb{R}$. Show that for every $x \geq 1$,

$$\sqrt{f(x)} \geq \sqrt{f(1)} + \frac{1}{2}(x - 1).$$

*Solution.* The derivative of $\sqrt{x}$ is $\frac{1}{2} x^{1/2-1} = 1/2\sqrt{x}$. So, $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{f'(x)}{2\sqrt{f(x)}}$. Now we can proceed in many ways. One way is to say that the function

$$g(x) = \sqrt{f(x)} - \frac{1}{2} x$$

has derivative

$$g'(x) = \frac{f'(x)}{2\sqrt{f(x)}} - \frac{1}{2} \geq 0,$$

hence $g$ is increasing and therefore for any $x \geq 1$, we have $g(x) \geq g(1)$, which gives the desired inequality. □

Another way: for any $t \geq 1$, we have

$$\frac{f'(t)}{2\sqrt{f(t)}} \geq \frac{1}{2}$$

which implies that

$$\int_1^x \frac{1}{2} dt \leq \int_1^x \frac{f'(t)}{2\sqrt{f(t)}} dt = \int_1^x \left(\sqrt{f(t)}\right)' dt = \sqrt{f(x)} - \sqrt{f(1)}.$$

which gives us the desired inequality. □

10. Let $f : [1, \infty) \to \mathbb{R}$ be a function satisfying $f(1) = 1$, and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

for every $x \geq 1$. Prove that $\lim_{x \to \infty} f(x)$ exists and this limit is less than $1 + \pi/4$.

*Solution.* First note that $f'(x) > 0$ so $f$ is increasing. Hence for $x \geq 1$, we can say that $f(x) \geq f(1) = 1$. Therefore,

$$f'(x) = \frac{1}{x^2 + f(x)^2} \leq \frac{1}{x^2 + 1} \text{ for all } x \geq 1. \tag{4}$$

Now

$$f(x) - f(1) = \int_1^x f'(t) dt \leq \int_1^x \frac{1}{1 + t^2} dt = \tan^{-1} x - \tan^{-1} 1 < \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Since $f$ is increasing and bounded above, we can say that $\lim_{x \to \infty} f(x)$ exists, and from the above inequalities, it is immediate that the limit should be less than or equal to $\pi/4$.

But how to claim that the limit is strictly less than $\pi/4$? Showing that is quite tricky, because even if you have $f(x) < g(x)$ for all $x$, taking limit as $x \to \infty$ (or $x \to a$) would change the $<$ sign to a $\leq$ sign. Here we adopt the following approach.

If $f$ never crosses $c$ where $1 < c < 1 + \pi/4$ then it is trivial that $\lim_{x \to \infty} f(x) \leq c < 1 + \pi/4$.

Else, $f(x_0) > c$ for some $x_0 > 1$, then $f(x) \geq f(x_0) > c$ for all $x > x_0$, and hence

$$f'(t) = \frac{1}{t^2 + f(t)^2} \leq \frac{1}{t^2 + c^2}, \quad \text{for } t \geq x_0.$$

Integrating this inequality from $x_0$ to $x$ and integrating (4) from 1 to $x_0$, we obtain

$$f(x) - f(1) \leq \int_1^{x_0} \frac{1}{t^2 + 1} dt + \int_{x_0}^x \frac{1}{t^2 + c^2} dt$$

for every $x > x_0$. Letting $x \to \infty$ here, we get

$$\lim_{x\to\infty} f(x) \leq 1 + \int_1^{x_0} \frac{1}{1+t^2}dt + \int_{x_0}^{\infty} \frac{1}{t^2+c^2}dt < 1 + \int_1^{\infty} \frac{1}{t^2+1}dt = 1 + \frac{\pi}{4}.$$

11. Let $f(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. For every $n \in \mathbb{N}$ let $f^n$ denote $f$ composed $n$-times, i.e., $f^{[n]}(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x)$. Evaluate $\int_0^1 f^{2020}(x)dx$.

*Solution.* First observe that $f(x) + f(1-x) = 1$ for every $x \in \mathbb{R}$. Then note that

$$f(f(1-x)) = f(1 - f(x)) = 1 - f(f(x)).$$

In fact, you can do induction on $n$ to show that if $g$ be $f$ composed with itself $n$ times, then $g$ also satisfies $g(x) + g(1-x) = 1$. Hence, for any $n \geq 1$, we can write

$$I = \int_0^1 f^{[n]}(x)\, dx = \int_0^1 f^{[n]}(1-x)\, dx = \int_0^1 \left(1 - f^{[n]}(x)\right) dx$$

and then add up these two alternate expressions for $I$ to show that $I = 1/2$. (Ans)

12. Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous. Define $a_n = \int_0^1 f(x+n)dx$, for every $n \geq 0$. Suppose also that $\lim_{n\to\infty} a_n = a$. Find the limit

$$\lim_{n\to\infty} \int_0^1 f(nx)dx.$$

*Solution.* We observe that

$$\int_0^1 f(nx)dx = \frac{1}{n}\int_0^n f(y)dy = \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1} f(y)dy = \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 f(u+k)du = \frac{1}{n}\sum_{k=0}^{n-1} a_k.$$

Now you have to use the following fact: if $(a_n)_{n\geq 0}$ be a sequence that converges to $a$, then the sequence $(b_n)_{n\geq 1}$ defined by

$$b_n = \frac{1}{n}\sum_{k=0}^{n-1} a_k$$

also converges to $a$. This tells us that the desired limit also equals $a$. (Ans)

Do you recall how to prove the fact used in the above proof? We just have to write

$$|b_n - a| = \left|\frac{1}{n}\sum_{k=0}^{n-1}(a_k - a)\right| \leq \frac{1}{n}\sum_{k=0}^{n-1}|(a_k - a)|$$

and truncate the sum at $N$ where $N$ is such that $|a_k - a| < \varepsilon/2$ holds for every $k \geq N$. Then we would have

$$|b_n - a| \leq \frac{1}{n}\sum_{k=0}^{N-1} |(a_k - a)| + \frac{1}{n}\sum_{k=N}^{n-1} |(a_k - a)| \leq \frac{B}{N} + \frac{n-N}{n}\frac{\varepsilon}{2}$$

where $B = \sum_{k=0}^{N-1} |a_k - a|$. It then follows that taking $n$ large enough so that $B/n < \varepsilon/2$ also holds, one obtains $|b_n - a| < \varepsilon$ for all sufficiently large $n$, which completes the proof.

13. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Prove that,

$$\lim_{n\to\infty} \int_a^b f(x)\sin(nx)dx = 0.$$

*Solution.* Applying integration by parts, we get

$$\begin{aligned}\int_a^b f(x)\sin(nx)dx &= \left[f(x)\int \sin(nx)dx\right]_a^b - \int_a^b \left(f'(x)\int \sin(nx)dx\right)dx \\ &= \frac{f(a)\cos na - f(b)\cos nb}{n} - \frac{1}{n}\int_a^b f'(x)\cos(nx)dx. \qquad (\dagger)\end{aligned}$$

Now, since $f$ is continuously differentiable on $[a, b]$, we can say that $f'$ is bounded on $[a, b]$. In other words, we can find an $M > 0$ such that $|f'(x)| < M$ holds for every $x \in [a, b]$. So, $0 \leq |f'(x)\cos nx| \leq M$ also holds for $x \in [a, b]$ and therefore we obtain from ($\dagger$) that

$$\begin{aligned}0 \leq \left|\int_a^b f(x)\sin(nx)dx\right| &\leq \left|\frac{f(a)\cos na - f(b)\cos nb}{n}\right| + \left|\frac{1}{n}\int_a^b f'(x)\cos(nx)dx\right| \\ &\leq \frac{|f(a)\cos na| + |f(b)\cos nb|}{n} + \frac{1}{n}\int_a^b |f'(x)\cos(nx)|dx \\ &\leq \frac{|f(a)| + |f(b)|}{n} + \frac{M(b-a)}{n} \to 0, \text{ as } n \to \infty.\end{aligned}$$

This proves that the desired limit is 0. ☐

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