Exercise 2 on Integration

1. (a) For each positive integer n, define a function f_n on $[0,1]$ by $f_n(x) = x^n$. Evaluate

$$\lim_{n\to\infty}\Big(\int_0^1 f_n(x)dx\Big) \text{ and } \int_0^1 \Big(\lim_{n\to\infty} f_n(x)\Big)dx.$$

   (b) Repeat the above exercise with $f_n(x) = nx^n$ for $0 \le x < 1$, and $f_n(1) = 0$.

2. Suppose that f has an anti-derivative F on an interval I, i.e. $F'(x) = f(x)$ holds for all $x \in I$. Let $x_0 \in I$ such that $\lim_{x\to x_0+} f(x) = a$. Prove that $f(x_0) = a$.

3. Let f be continuous on $[a,b]$. Suppose that $\displaystyle\int_a^c f(x)dx = 0$ holds for every $a \le c \le b$. Show that f must be identically zero on $[a,b]$.

4. Let f be continuous on $\mathbb{R}$. Suppose that for some $T > 0$,

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$$

   holds for every $a \in \mathbb{R}$. Show that $f(x+T) = f(x)$ for every $x \in \mathbb{R}$.

5. Define $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{x^2+1}}dx$, for every $n \in \mathbb{N}$. Prove that, $\displaystyle\lim_{n\to\infty} nI_n = \frac{1}{\sqrt{2}}$.

6. Prove the inequality: $0.4 < \displaystyle\int_0^1 x^{\sin x+\cos x}dx < 0.5$.

7. Find $a \in \mathbb{R}$ which maximises the value of the following integral

$$\int_{a-1}^{a+1} \frac{1}{1+x^8}dx.$$

8. Let $f : [0,\infty) \to \mathbb{R}$ be a non-decreasing function. Then show that the following inequality holds for all x, y, z such that $0 \le x < y < z$.

$$(z-x)\int_y^z f(u)du \ge (z-y)\int_x^z f(u)du.$$

9. Let $f(u)$ be a continuous function and, for any real number u, let $[u]$ denote the greatest integer less than or equal to u. Show that for any $x > 1$,

$$\int_1^x [u]([u]+1)f(u)du = 2\sum_{n=1}^{[x]} n\int_n^x f(u)du.$$

Solutions to Exercise 2

1. (a) For each positive integer n, define a function f_n on $[0,1]$ by $f_n(x) = x^n$. Evaluate

$$\lim_{n\to\infty}\Big(\int_0^1 f_n(x)dx\Big) \text{ and } \int_0^1\Big(\lim_{n\to\infty} f_n(x)\Big)dx.$$

(b) Repeat the above exercise with $f_n(x) = nx^n$ for $0 \le x < 1$, and $f_n(1) = 0$.

Solution.

(a) This is really straightforward. On one hand we have

$$\lim_{n\to\infty}\Big(\int_0^1 f_n(x)dx\Big) = \lim_{n\to\infty}\Big(\int_0^1 x^n dx\Big) = \lim_{n\to\infty}\frac{1}{n+1} = 0,$$

while on the other hand we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Hence $\displaystyle\int_0^1\Big(\lim_{n\to\infty} f_n(x)\Big)dx = 0.$

(b) First note that

$$\lim_{n\to\infty}\Big(\int_0^1 f_n(x)dx\Big) = \lim_{n\to\infty}\Big(\int_0^1 nx^n dx\Big) = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

Next, we show that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx^n = 0, \text{ for every } 0 \le x \le 1.$$

This is trivial for $x = 0$ and $x = 1$ (since $f_n(1) = 0$ for all $n \ge 1$). For $0 < x < 1$, we take the help of $r = 1/x > 1$, and see that

$$0 < nx^n = \frac{n}{(1+r-1)^n} \le \frac{n}{\binom{n}{2}(r-1)^2} = \frac{2}{(r-1)^2(n-1)} \to 0 \text{ (as } n\to\infty).$$

Therefore in this problem we have

$$\int_0^1\Big(\lim_{n\to\infty} f_n(x)\Big)dx = 0 \ne 1 = \lim_{n\to\infty}\Big(\int_0^1 f_n(x)dx\Big).$$

2. Suppose that f has an anti-derivative F on an interval I, i.e. $F'(x) = f(x)$ holds for all $x \in I$. Let $x_0 \in I$ such that $\displaystyle\lim_{x\to x_0+} f(x) = a$. Prove that $f(x_0) = a$.

Solution. For any $x > x_0$, we apply MVT on F to say that there exists $c_x \in (x_0, x)$ such that

$$\frac{F(x) - F(x_0)}{x - x_0} = F'(c_x) = f(c_x). \tag{2}$$

Now letting $x \to x_0^+$, the above LHS converges to $F'(x_0) = f(x_0)$. What about the RHS? Since $x_0 < c_x < x \implies c_x \to x_0^+$, so the RHS of (2) converges to $\lim_{x \to x_0^+} f(c_x) = \lim_{c_x \to x_0^+} f(c_x) = a$. Hence we get the desired conclusion. $\square$

3. Let f be continuous on $[a, b]$. Suppose that $\displaystyle\int_a^c f(x)dx = 0$ holds for every $a \le c \le b$. Show that f must be identically zero on $[a, b]$.

   Solution. One way to attack this problem is by means of contradiction[3]. But a simpler way is to use FTC, as follows. Define $F(x) = \int_a^x f(t)dt$ for $x \in [a, b]$. Since f is continuous, it holds by FTC that F is differentiable and $F' = f$ on (a, b). But it is given that F is a constant function, implying that $f(x) = F'(x) = 0$ for any $x \in (a, b)$. Finally, continuity ensures that f must also vanish at the endpoints of $[a, b]$. $\square$

4. Let f be continuous on $\mathbb{R}$. Suppose that for some $T > 0$,

   $$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$$

   holds for every $a \in \mathbb{R}$. Show that $f(x + T) = f(x)$ for every $x \in \mathbb{R}$.

   Solution. Define $F(x) = \int_a^x f(t)dt$ for $x \in \mathbb{R}$. Since f is continuous, it holds by FTC that F is differentiable and $F' = f$ on $\mathbb{R}$. Now,

   $$g(a) \overset{\text{def}}{=} \int_a^{a+T} f(t)dt = F(a + T) - F(a)$$

   is given to be a constant function (since $g(a) = g(0)$ for every $a \in \mathbb{R}$). Hence for every $a \in \mathbb{R}$, we must have $g'(a) = 0$. But $g'(a) = F'(a + T) - F'(a) = f(a + T) - f(a)$. So we get the desired conclusion that $f(a + T) = f(a)$ must hold for every $a \in \mathbb{R}$. $\square$

5. Define $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{x^2 + 1}}dx$, for every $n \in \mathbb{N}$. Prove that, $\displaystyle\lim_{n\to\infty} nI_n = \frac{1}{\sqrt{2}}$.

   Solution. Since $x^2 + 1 \ge 2x > 0$, we have

   $$I_n = \int_0^1 \frac{x^n}{\sqrt{x^2 + 1}}dx \le \int_0^1 \frac{x^n}{\sqrt{2x}}dx = \frac{1}{\sqrt{2}(n + 1/2)}.$$

   Can we also give a similar lower bound so that Sandwich principle can be applied?

   Giving such a lower bound turns out be actually easier. Note that $0 \le x \le 1 \implies x^2 + 1 \le 2$,

[3] You should try this. Write an alternate proof using $\varepsilon - \delta$'s which ultimately gives you a contradiction.

which gives the following lower bound:

$$I_n = \int_0^1 \frac{x^n}{\sqrt{x^2+1}}dx \geq \int_0^1 \frac{x^n}{\sqrt{2}}dx = \frac{1}{\sqrt{2}(n+1)}.$$

Combining the above two inequalities and invoking the sandwich principle, we can conclude that

$$\lim_{n\to\infty} nI_n = \frac{1}{\sqrt{2}}.$$

□

6. Prove the inequality: $0.4 < \int_0^1 x^{\sin x+\cos x}dx < 0.5$.

Solution. First recall that $1 < \sin x + \cos x < \sqrt{2}$ for every $x \in (0, \pi/2)$. Also recall that, when $x \in (0,1)$, $a > b$ actually implies $x^a < x^b$ (not its opposite). Therefore,

$$\forall x \in (0,1],\ x^{\sqrt{2}} < x^{\sin x+\cos x} < x^1.$$

(At $x = 0$ these are all equal.) Upon integration, the above inequalities produce the following:

$$\int_0^1 x^{\sqrt{2}}dx < \int_0^1 x^{\sin x+\cos x}dx < \int_0^1 x^1 dx.$$

(Do you see why strict inequality holds here?) Finally note that $\int_0^1 x^{\sqrt{2}}dx = \sqrt{2}-1 > 0.4$. Thus we get the desired inequality

$$0.4 < \int_0^1 x^{\sqrt{2}}dx < \int_0^1 x^{\sin x+\cos x}dx < \int_0^1 x^1 dx = 0.5.$$

□

7. Find $a \in \mathbb{R}$ which maximises the value of the following integral

$$\int_{a-1}^{a+1} \frac{1}{1+x^8}dx.$$

Solution. We start by defining $I(a) = \int_{a-1}^{a+1} \frac{1}{1+x^8}dx$. Applying the Leibniz rule, we get

$$I'(a) = \frac{1}{1+(a+1)^8} - \frac{1}{1+(a-1)^8}.$$

Note that, $I'(a) = 0 \iff (a+1)^8 = (a-1)^8 \iff a = 0, \frac{1}{2}$. Now $I'(a)$ is of the form

$$I'(a) = c(a)\cdot((a-1)^2 - (a+1)^2) = c(a)\cdot(-4a),$$

where $c(a)$ is positive for any $a \in \mathbb{R}$. Therefore, $I'(a)$ changes its sign only when $a = 0$.

Furthermore, note that $I'(a) > 0$ for $a < 0$ and $I'(a) < 0$ for $a > 0$. Hence $I(a)$ is maximised at $a = 0$. (Ans)

8. Let $f : [0, \infty) \to \mathbb{R}$ be a non-decreasing function. Then show that the following inequality holds for all x, y, z such that $0 \leq x < y < z$.

$$(z-x)\int_y^z f(u)du \geq (z-y)\int_x^z f(u)du.$$

Solution. If f had an anti-derivative, the problem would have been much easier. How? Suppose that $F' = f$. Since $F' = f$ is non-decreasing, F must be convex. Also, $\int_a^b f(u)du = F(b) - F(a)$ for any $a, b \geq 0$. Hence, the given inequality simplifies as follows:

$$\begin{aligned} & (z-x)\int_y^z f(u)du \geq (z-y)\int_x^z f(u)du \\ \iff & (z-x)(F(z)-F(y)) \geq (z-y)(F(z)-F(x)) \\ \iff & \frac{F(z)-F(y)}{z-y} \geq \frac{F(z)-F(x)}{z-x} \\ \iff & F(y) \leq \frac{z-y}{z-x}F(x) + \frac{y-x}{z-x}F(z) \end{aligned} \tag{$*$}$$

Now observe that $y = \lambda x + (1-\lambda)z \iff \lambda = (z-y)/(z-x)$. So $(*)$ is same as saying

$$F(\lambda x + (1-\lambda)z) \leq \lambda F(x) + (1-\lambda)F(z)$$

which follows from the convexity of F. Another way to finish the above solution is as follows. Observe that

$$\frac{F(z)-F(y)}{z-y} \geq \frac{F(z)-F(x)}{z-x} \iff \frac{F(z)-F(y)}{z-y} \geq \frac{F(y)-F(x)}{y-x}.$$

Now by MVT, there exists $c_1 \in (x, y)$ and $c_2 \in (y, z)$ such that

$$\frac{F(y)-F(x)}{y-x} = F'(c_1) = f(c_1), \text{ and } \frac{F(z)-F(y)}{z-y} = F'(c_2) = f(c_2).$$

Since $c_1 < y < c_2$ and f is non-decreasing, we get $f(c_1) \leq f(c_2)$, which completes the proof.

Next, let us discuss a proof that do not rely on the assumption that f has an anti-derivative.

$$
\begin{aligned}
&(z-x)\int_y^z f(u)du \geq (z-y)\int_x^z f(u)du \\
\iff &(z-x)\int_y^z f(u)du \geq (z-y)\int_x^y f(u)du + (z-y)\int_y^z f(u)du \\
\iff &(y-x)\int_y^z f(u)du \geq (z-y)\int_x^y f(u)du
\end{aligned}
$$

Since f is non-decreasing, for any $u \in (y,z)$ we have $f(u) \geq f(y)$ and for any $u \in (x,y)$ we have $f(u) \leq f(y)$. Therefore,

$$
\begin{aligned}
(y-x)\int_y^z f(u)du \geq (y-x)\int_y^z f(y)du &= (y-x)(z-y)f(y) \\
&= \int_x^y f(y)du \geq (z-y)\int_x^y f(u)du.
\end{aligned}
$$

This completes the proof. □

9. Let $f(u)$ be a continuous function and, for any real number u, let $[u]$ denote the greatest integer less than or equal to u. Show that for any $x > 1$,

$$\int_1^x [u]([u]+1)f(u)du = 2\sum_{n=1}^{[x]} n\int_n^x f(u)du.$$

Solution. Let $[x] = m$. The LHS divided by 2 can be written as

$$\int_1^x \frac{[u]([u]+1)}{2} f(u)du = \sum_{k=1}^{m-1} \frac{k(k+1)}{2}\int_k^{k+1} f(u)du + \frac{m(m+1)}{2}\int_m^x f(u)du. \tag{3}$$

On the other hand, the RHS divided by 2 can be simplified as

$$\sum_{n=1}^{[x]} n\int_n^x f(u)du = \int_1^x f(u)du + 2\int_2^x f(u)du + \cdots + m\int_m^x f(u)du \tag{4}$$

Now we can break each integral on the RHS of (4) as sum of 'consecutive' integrals, e.g.,

$$\int_1^x f(u)du = \int_1^2 f(u)du + \int_2^3 f(u)du + \cdots + \int_{m-1}^m f(u)du + \int_m^x f(u)du.$$

In this manner, note that for any $1 \leq k \leq m-1$, the integral $\int_k^{k+1} f(u)du$ appears on the RHS of (4) exactly $(1+2+\cdots+k) = k(k+1)/2$ times, while the integral $\int_m^x f(ud)du$ appears $m(m+1)/2$ times. Therefore, the RHS of (4) is same as the RHS of (3) which completes the proof. □

Alternate proof. *(of the fact that the RHS of (4) and (3) are the same)*.

$$\begin{aligned}
\sum_{n=1}^{[x]} n \int_n^x f(u)du &= \sum_{1\le n\le m} \ \sum_{n\le k\le m-1} \int_k^{k+1} nf(u)du + \sum_{1\le n\le m} \int_m^x nf(u)du \\
&= \sum_{1\le k\le m-1} \ \sum_{n\le k} \int_k^{k+1} nf(u)du + \frac{m(m+1)}{2} \int_m^x nf(u)du \\
&= \sum_{1\le k\le m-1} \frac{k(k+1)}{2} \int_k^{k+1} nf(u)du + \frac{m(m+1)}{2} \int_m^x nf(u)du.
\end{aligned}$$

Although the above proof uses less words (and more symbols), some students might find it rather difficult to understand. The key idea in this proof is the swapping of the order of summation. Think of the points $\{(n,k) : 1 \le n \le m, n \le k \le m-1\}$ which are just a bunch of lattice points arranged as a triangular array in the 2D-plane, with n varying along the x-axis and k along the y-axis. Initially we were fixing each column and first sum over the column (i.e., sum over k) and then add up these column-sums (i.e., sum over n). This should be same as first summing over each row (i.e., first sum over n) and then add up these row-sums (i.e., sum over k). That's the trick!
