Integration : Theory and Problems (Day 2)

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1 Connection with derivatives

It would not be much of an overstatement if we say that derivatives and integrals are the two main pillars of the whole of Calculus. Derivatives represent the rate of change of a function and integrals represent the area under the curves. In this section we shall try to find the connection between these two seemingly different concepts.

Suppose that f is differentiable on $[a, b]$. Let us take a partition of $[a, b]$, say $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$. Observe that we can write

$$f(b) - f(a) = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) = \sum_{i=1}^{n} \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \cdot (x_i - x_{i-1}). \tag{1}$$

Now Lagrange's Mean Value Theorem tells us that for each sub-interval $[x_{i-1}, x_i]$, there exists $t_i \in (x_{i-1}, x_i)$ such that

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(t_i).$$

Therefore we can write from (1) that

$$\sum_{i=1}^{n} f'(t_i) \cdot (x_i - x_{i-1}) = f(b) - f(a).$$

If we now assume that f' *is integrable*, then making the partition finer and finer makes the above LHS closer and closer to

$$\int_a^b f'(x)dx.$$

In fact, Theorem 1.3 applies here and tells us that the LHS converge to $\displaystyle\int_a^b f'(x)dx$ if the maximum length of any sub-interval for P goes to zero.

Hence we have the following theorem.

Theorem 1.1 (FTC-integral of a derivative)**.**
If f is differentiable on $[a, b]$ such that f' is integrable on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

The above theorem is known as *a* Fundamental Theorem of Calculus. It tells us what will be the integral of a derivative. There is one more fundamental theorem (given below) which tells us what will be the derivative of an integral.

The importance of the above theorem lies in the fact that it allows us to calculate the integral of a function g provided we have another function f such that $f' = g$. Such a function f is called an anti-derivative of g. Why 'an'? Because it is not unique: if $f(x)$ is an anti-derivative of $g(x)$, then so is $f(x) + c$, for any constant c. Does integrable functions always have anti-derivatives? In general the answer is 'No' (we'll see such examples). However, if we impose continuity, then we get a positive answer, as given in the following theorem.

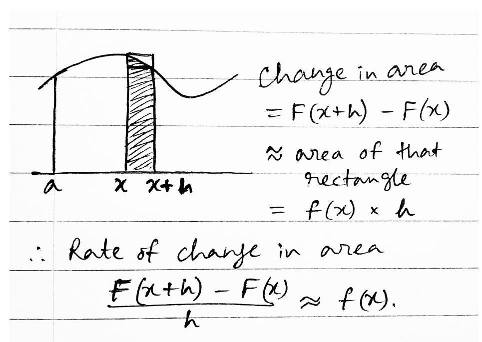
Theorem 1.2 (FTC-derivative of an integral)**.**
Suppose that f is integrable on $[a, b]$. Define

$$F(x) = \int_a^x f(t)dt, \text{ for } a \leq x \leq b.$$

Then, (i) F is continuous on $[a, b]$, and (ii) if f is continuous at $c \in [a, b]$, then F will be differentiable at c, with $F'(c) = f(c)$.

Before proving this theorem, let us understand the what it really says and why that should be true. The theorem says that if we 'slide' x from a to b, the (signed) area $\int_a^x f(t)dt$ changes continuously (as a function of x) and if f is continuous at c, then the rate of this change in area (at c) is same as value of f at that point[1].

However, it might not be very clear why the rate of the change in area is same as the value of f at that point. For this, consider the diagram on the right. If we take h small enough, then the change in area from x to $x + h$, which is $F(x + h) - F(x)$, can be very well approximated by the area of the rectangle that has height $f(x)$ and width h. Hence, the rate of change in area is approximately $f(x)$. We shall next prove (rigorously) that as $h \to 0$, this rate of change converges to $f(x)$, which is exactly what the theorem tells us.



Proof of Theorem 1.2. Since integrals are defined for bounded functions only, let us take $M > 0$ such that $|f(x)| < M$ holds for all $x \in [a, b]$. Then, for any $x \in [a, b]$ and $h > 0$ (such that $x + h \in [a, b]$ as well) we have

$$|F(x + h) - F(x)| = \left| \int_x^{x+h} f(t)dt \right| \leq \int_x^{x+h} |f(t)|\, dt \leq \int_x^{x+h} M dt = Mh.$$

[1] An animation is given here: https://www.desmos.com/calculator/7994k6zj6c. In this animation, there is a slider that changes the value of c (from a to b) and you can see how the area from a to c changes (continuously).

Similarly, for $h < 0$, we have

$$|F(x+h) - F(x)| \leq \int_{x+h}^{x} M dt = M(-h).$$

Combining these, we may write $|F(x+h) - F(x)| \leq M|h|$, which implies that F is (uniformly[2]) continuous on $[a, b]$.

To prove the other part, let f be continuous at c. Observe that for $h > 0$,

$$\begin{aligned}\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| &= \left|\frac{1}{h}\int_{c}^{c+h} f(t)dt - hf(c)\right| \\ &= \frac{1}{h}\left|\int_{c}^{c+h} (f(t) - f(c))dt\right| \\ &\leq \frac{1}{h}\int_{c}^{c+h} |f(t) - f(c)|dt.\end{aligned}$$

Now, since f is continuous at c, so for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t) - f(c)| \leq \varepsilon$ holds for every $t \in [a, b]$ such that $|t - c| \leq \delta$. Hence, if we take $0 < h < \delta$, then we have

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| \leq \frac{1}{h}\int_{c}^{c+h} |f(t) - f(c)|dt \leq \frac{\varepsilon}{h}\int_{c}^{c+h} dt = \varepsilon.$$

Similar result holds for $h < 0$. Combining them, we can say that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| < \varepsilon \text{ holds for } 0 < |h| < \delta.$$

Therefore, from the definition of derivative, we conclude that $F'(c) = f(c)$. $\square$

When I first learned the above two theorems (Theorems 1.1 and 1.2), I had the following thought:

> If f is continuous and I want to find $\int_a^b f(x)dx$, then Theorem 1.1 says that I need an anti-derivative F of f, and Theorem 1.2 provides one such anti-derivative. So in this manner, I can find $\int_a^b f(x)dx$ for any continuous function f!

Unfortunately, this is a very stupid idea. To see why, take any continuous f, say $f(x) = x^2$. Then, note that the anti-derivative of $f(x) = x^2$ provided by Theorem 1.2 is

$$F(x) = \int_{0}^{x} tdt$$

[2]Here uniform means the δ depends only upon ε, not on x. See the note on uniform continuity here.

and then Theorem 1.1 would help us calculate the integral as

$$\int_1^2 x dx = F(2) - F(1).$$

But how to get the values of $F(2)$ and $F(1)$? The way F is defined, we have to calculate an integral which is essentially same as the original one – we are back to square one!

Thus, although Theorem 1.2 guarantees the existence of an anti-derivative of any continuous function, it does not provide us a way to calculate it. And without an anti-derivative how can we find the integral using Theorem 1.1!

This is why students are first taught how to guess an anti-derivative, under the name of *'indefinite integration'*. All the chapters on indefinite integration actually teach us how to cleverly *guess* an anti-derivative for a given function. For example, we learn that an anti-derivative of x is $x^2/2$, and we write

$$\int x dx = x^2/2 + c,$$

to mean that a function F is an anti-derivative of $f(x) = x$ if and only if $F(x) = x^2/2 + c$ for some constant c. The notation

$$\int x \, dx$$

is simply a placeholder for 'an anti-derivative of $f(x) = x$', it does not directly relate to an area in any way. From a computational perspective, finding an anti-derivative for a given function g is not easy (at least not as easy as finding the derivative of a function). That is why we develop different ways to guess anti-derivatives, e.g., substitution, integration by parts, reduction formulae etc. I assume that the reader has already seen a fair amount of indefinite integration (or, will see them in due course of time), here we shall mainly focus on the theory of (definite) integrals that relate to area under the curves!

The following examples illustrate why one must check the conditions before applying the above theorems.

Example 1.1. Define $f(x) = x^2 \sin \frac{1}{x^2}$ if $x \neq 0$ and $f(0) = 0$. Note that f is differentiable on $[-1, 1]$. For $x \neq 0$,

$$f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$$

and $f'(0) = 0$. Yet, it is incorrect to write that

$$\int_{-1}^{1} f'(x) dx = f(1) - f(-1) = 0.$$

Why? Because f' being unbounded near $x = 0$, the above integral is an improper integral, so we cannot directly apply FTC here. In fact, it turns out that the above integral does not exist, even in the improper sense. To see why, let $g(x) = \frac{2}{x}\cos(1/x^2)$ and note that $\cos(1/x^2) \geq 1/2$ for $|x| < \sqrt{3/\pi} = a$, hence

$$\int_0^a \frac{2}{x}\cos\left(\frac{1}{x^2}\right)dx \geq \int_0^a \frac{1}{x}dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^a \frac{1}{x}dx = \lim_{\varepsilon\to 0^+}(\log a - \log\varepsilon) = +\infty.$$

Similarly you can show that $\int_{-a}^0 g(x)dx = -\infty$. Hence $\int_{-a}^a g(x)dx$ does not exist, even though it is tempting to write $\int_{-a}^a g(x)dx = \int_0^a (g(t)+g(-t))dt = 0$.

Example 1.2. Take $f(x) = \lfloor x \rfloor$, $0 \leq x \leq 2$. This function has discontinuity only at $x = 1$ and $x = 2$, so it is integrable. Define

$$F(x) = \int_0^x f(t)dt,\ 0 \leq x \leq 2.$$

Observe that $F(x) = 0$ for $0 \leq x \leq 1$ and $F(x) = x - 1$ for $1 \leq x \leq 2$. Hence, F is not differentiable at $x = 1$. Does it contradict FTC? No, because f is not continuous at $x = 1$.

(In fact, this function f cannot have an anti-derivative. Because, if there were any function F such that $F' = f$, then f must have the Intermediate Value Property (derivatives always have IVP). But our f does not have IVP.)

Question 1. *In Example 1, f is differentiable but the derivative is not integrable, while in Example 2, f is integrable but the integral is not differentiable. Now, if f is integrable on $[a,b]$ and the integral from a to x is differentiable w.r.t. x, is it necessary that F' equals f?*

The answer to the above question is again in the negative, as shown by the following example.

Example 1.3. Consider $f : [0,2] \to \mathbb{R}$ as $f(x) = x$ for $x \neq 1$, and set $f(1) = 2$. Define $F(x) = \int_0^x f(t)dt,\ 0 \leq x \leq 2$. Clearly, F is differentiable, but $F'(1) \neq f(1)$.

Theorem referred from previous class:

Theorem 1.3. *If f is integrable on $[a,b]$, then it holds that*

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\cdot\frac{b-a}{n}\right) = \int_a^b f(x)dx.$$

2 Some example problems

Problem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define $g(x) = \int_0^x (x-t)f(t)dt$ for every $x \in \mathbb{R}$. Show that $g'' = f$.*

Solution. First we write $g(x)$ as:

$$g(x) = x\int_0^x f(t)\ dt - \int_0^x tf(t)\ dt.$$

Since f is continuous on $\mathbb{R}$, by the fundamental theorem of calculus (FTC) we know that $F(x) = \int_0^x f(t)\ dt$ is differentiable at each $x \in \mathbb{R}$ and $F' = f$. Also, since $h(t) = tf(t)$ is continuous on $\mathbb{R}$, by FTC we know that

$$H(x) = \int_0^x tf(t)\ dt$$

is differentiable at each $x \in \mathbb{R}$ and $H'(x) = h(x) = xf(x)$. Therefore $g(x) = xF(x) - H(x)$ is differentiable and

$$g'(x) = F(x) + xf(x) - xf(x) = F(x).$$

Hence g' is also differentiable and $g''(x) = F'(x) = f(x)$. □

Problem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that for every $a \in \mathbb{R}$,*

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx.$$

Solution. Define $F(x) = \int_0^x f(t)dt$ for any $x \in \mathbb{R}$. Also, let

$$g(a) = \int_a^{a+T} f(x)dx = F(a+T) - F(a).$$

Since f is continuous on $\mathbb{R}$, by FTC, we know that $F'(x) = f(x)$ at every $x \in \mathbb{R}$. Hence

$$\frac{d}{da}g(a) = \frac{d}{da}(F(a+T) - F(a)) = f(a+T) - f(a) = 0.$$

Hence g is a constant function, and therefore $g(a) = g(0)$, i.e., for every $a \in \mathbb{R}$,

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$$

which is exactly what we had to show. □

Problem 2.3. *(Leibniz Rule) Let f be continuous on $[a, b]$, and u and v be differentiable functions from $[c, d]$ to $[a, b]$. Prove that,*

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x).$$

Solution. Define $F(y) = \displaystyle\int_a^y f(t)dt$ for every $y \in [a, b]$. Now for $x \in [c, d]$,

$$I(x) = \int_{u(x)}^{v(x)} f(t)dt = F(v(x)) - F(u(x)).$$

Since f is continuous, we know that $F' = f$. Hence, by chain rule of differentiation,

$$I'(x) = F'(v(x))v'(x) - F'(u(x))u'(x) = f(v(x))v'(x) - f(u(x))u'(x)$$

for every $x \in [c, d]$. □

Problem 2.4. *Let $a_0 = 0 < a_1 < a_2 < \cdots < a_n$ be real numbers. Suppose that $p(t)$ is a real valued polynomial of degree n such that*

$$\int_{a_j}^{a_{j+1}} p(t)\ dt = 0, \quad \textit{for each } 0 \leq j \leq n-1.$$

Prove that the polynomial $p(t)$ must have exactly n real roots.

Solution. Define $F(x) = \displaystyle\int_0^x p(t)dt$, for any $x > 0$. For each $0 \leq j \leq n-1$,

$$\int_{a_j}^{a_{j+1}} p(t)\ dt = F(a_{j+1}) - F(a_j) = 0.$$

Since $p(t)$ is continuous everywhere, we have $F'(x) = p(x)$. By applying Rolle's theorem on $F(x)$, we get a root of $p(x) = F'(x)$ in each (a_j, a_{j+1}). □

Exercise 2 on Integration

1. (a) For each positive integer n, define a function f_n on $[0,1]$ by $f_n(x) = x^n$. Evaluate

$$\lim_{n\to\infty}\Big(\int_0^1 f_n(x)dx\Big) \text{ and } \int_0^1 \Big(\lim_{n\to\infty} f_n(x)\Big)dx.$$

   (b) Repeat the above exercise with $f_n(x) = nx^n$ for $0 \le x < 1$, and $f_n(1) = 0$.

2. Suppose that f has an anti-derivative F on an interval I, i.e. $F'(x) = f(x)$ holds for all $x \in I$. Let $x_0 \in I$ such that $\lim\limits_{x\to x_0+} f(x) = a$. Prove that $f(x_0) = a$.

3. Let f be continuous on $[a,b]$. Suppose that $\displaystyle\int_a^c f(x)dx = 0$ holds for every $a \le c \le b$. Show that f must be identically zero on $[a,b]$.

4. Let f be continuous on $\mathbb{R}$. Suppose that for some $T > 0$,

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$$

   holds for every $a \in \mathbb{R}$. Show that $f(x+T) = f(x)$ for every $x \in \mathbb{R}$.

5. Define $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{x^2+1}}dx$, for every $n \in \mathbb{N}$. Prove that, $\lim\limits_{n\to\infty} nI_n = \dfrac{1}{\sqrt{2}}$.

6. Prove the inequality: $0.4 < \displaystyle\int_0^1 x^{\sin x + \cos x}dx < 0.5$.

7. Find $a \in \mathbb{R}$ which maximises the value of the following integral

$$\int_{a-1}^{a+1} \frac{1}{1+x^8}dx.$$

8. Let $f : [0,\infty) \to \mathbb{R}$ be a non-decreasing function. Then show that the following inequality holds for all x, y, z such that $0 \le x < y < z$.

$$(z-x)\int_y^z f(u)du \ge (z-y)\int_x^z f(u)du.$$

9. Let $f(u)$ be a continuous function and, for any real number u, let $[u]$ denote the greatest integer less than or equal to u. Show that for any $x > 1$,

$$\int_1^x [u]([u]+1)f(u)du = 2\sum_{n=1}^{[x]} n\int_n^x f(u)du.$$
