Exercise 1 on Integration

1. Define $f(x) = \int_0^1 |t - x|t\,dt$, for $x \in \mathbb{R}$. Sketch the graph of $f(x)$. What is the minimum value of $f(x)$?

2. For any positive integer n, let $C(n)$ denote the number of points which have integer coordinates and lie inside the circle $x^2 + y^2 = n^2$. Show that the limit

$$\lim_{n\to\infty} \frac{C(n)}{n^2}$$

exists and also evaluate this limit. Can you explain the result intuitively?

3. Let f, g be polynomials of degree n such that $\int_0^1 x^k f(x)dx = \int_0^1 x^k g(x)dx$ holds for each $k = 0, 1, \ldots, n$. Show that $f = g$.

4. Let f, g be continuous and positive functions defined on $[0, 1]$ satisfying

$$\int_0^1 f(x)dx = \int_0^1 g(x)dx.$$

Define $y_n = \int_0^1 \frac{(f(x))^{n+1}}{(g(x))^n}dx$, for every integer $n \geq 0$. Show that $\{y_n\}_{n\geq 0}$ is an increasing sequence.

5. Suppose that f is integrable on $[a, b]$. Define

$$F(x) = \int_a^x f(t)dt, \text{ for } a \leq x \leq b.$$

Then, (i) F is continuous on $[a, b]$, and (ii) if f is continuous at $c \in [a, b]$, then F will be differentiable at c, with $F'(c) = f(c)$.

6. If f is differentiable on $[a, b]$ such that f' is continuous on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

7. If f is continuous on $[a, b]$, show that $\int_a^b f(t)dt = f(c)(b - a)$ must hold for some $c \in (a, b)$.

8. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)dx = 1$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = 3c^2$.

9. Prove the inequalities: $\frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx \leq \frac{2\pi^2}{9}$.

Solutions to Exercise 1

1. Let $f(x) = \displaystyle\int_0^1 |t - x| t \, dt$, defined for $x \in \mathbb{R}$. Sketch the graph of $f(x)$. What is the minimum value of $f(x)$?

   Solution. Note that for $0 \le x \le 1$, we have

$$f(x) = \int_0^1 |t - x| t \, dt = \int_0^x (x - t) t \, dt + \int_x^1 (t - x) t \, dt = \frac{x^3}{3} - \frac{x}{2} + \frac{1}{3}.$$

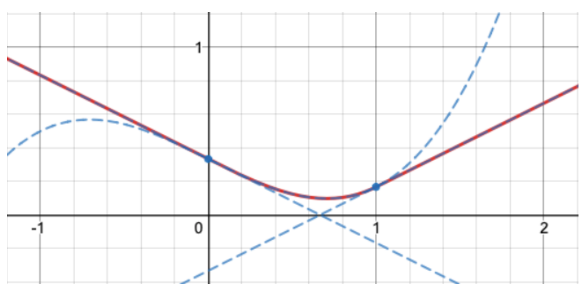
   For $x < 0$,

$$f(x) = \int_0^1 |t - x| t \, dt = \int_0^1 (t - x) t \, dt = \frac{t^3}{3} - x\frac{t^2}{2}\Big|_0^1 = \frac{1}{3} - \frac{x}{2}.$$

   Finally, for $x > 1$,

$$f(x) = \int_0^1 |t - x| t \, dt = \int_0^1 (x - t) t \, dt = x\frac{t^2}{2} - \frac{t^3}{3}\Big|_0^1 = \frac{x}{2} - \frac{1}{3}.$$

   It is quite easy to draw the graph of this function $f(x)$, because for $x \in [0, 1]$ it is just a cubic polynomial and for both $x \le 0$ and $x \ge 1$ it is a straight line.



   The minimum value can be determined by differentiating f. It turns out that f attains its global minimum at $x = 1/\sqrt{2}$, and the minimum value is $\frac{1}{3}(1 - \frac{1}{\sqrt{2}})$. □

2. For any positive integer n, let $C(n)$ denote the number of points which have integer coordinates and lie inside the circle $x^2 + y^2 = n^2$. Show that the limit

$$\lim_{n\to\infty} \frac{C(n)}{n^2}$$

   exists and also evaluate this limit. Can you explain the result intuitively?

   Solution. By symmetry, it is enough to find the number of lattice points (points having integer coordinates) in the first quadrant. This can be calculated by first fixing the x-coordinate to be k and then summing up for $k = 1, 2, \ldots, n$. Note that

$$\#\left\{(x, y) : x = k, y \in \mathbb{Z}, y \ge 0 \text{ and } x^2 + y^2 \le n^2\right\} = \lfloor\sqrt{n^2 - k^2}\rfloor.$$

Hence,

$$C(n) = 4\sum_{k=1}^{n}\lfloor\sqrt{n^2-k^2}\rfloor + 1$$

where the last $+1$ is for the origin $(0,0)$. Next, in order to calculate the limit of $C(n)/n^2$ as $n \to \infty$, observe that it is enough to calculate the limit of $n^{-2}\sum_{k=1}^{n}\lfloor\sqrt{n^2-k^2}\rfloor$ and we can handle the floor function using sandwich principle.The inequality $x-1 \le \lfloor x\rfloor \le x$ produces the following bounds

$$\frac{1}{n^2}\left(\sum_{k=1}^{n}\sqrt{n^2-k^2}-n\right) \le \frac{1}{n^2}\sum_{k=1}^{n}\lfloor\sqrt{n^2-k^2}\rfloor \le \frac{1}{n^2}\left(\sum_{k=1}^{n}\sqrt{n^2-k^2}\right).$$

Observe that

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n}\sqrt{n^2-k^2}-n\right) = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2} = \int_0^1\sqrt{1-x^2}\,dx.$$

This integral calculates the area of one quarter of the unit circle, hence equals $\pi/4$. (Alternately, you can use integration by parts.) Finally, applying the Sandwich theorem we conclude that

$$\lim_{n\to\infty}\frac{C(n)}{n^2} = \lim_{n\to\infty}\frac{4}{n^2}\sum_{k=1}^{n}\lfloor\sqrt{n^2-k^2}\rfloor = 4\times\frac{\pi}{4} = \pi.$$

This is intuitive, because the area of the circle $x^2+y^2 \le n^2$ being πn^2 (square units), it should include approximately πn^2 many unit squares. The above limit makes this idea precise.

3. Let f, g be polynomials of degree n such that $\int_0^1 x^k f(x)dx = \int_0^1 x^k g(x)dx$ holds for each $k = 0, 1, \ldots, n$. Show that $f = g$.

Solution. Since $h(x) = f(x) - g(x)$ is a polynomial of degree less than or equal to n, and h satisfies

$$\int_0^1 x^k h(x)dx = 0 \text{ for each } k = 0, 1, \ldots, n,$$

we can easily deduce that

$$\int_0^1 h(x)^2 dx = 0.$$

But $h(x)^2$ is a non-negative and continuous function, so the above equation can hold if and only if h is identically zero on $[0,1]$. Therefore, $f(x) = g(x)$ for every $x \in [0,1]$. Since f and g are polynomials, this is enough to conclude that $f = g$. □

4. Let f, g be continuous and positive functions defined on $[0,1]$ satisfying

$$\int_0^1 f(x)dx = \int_0^1 g(x)dx.$$

Define $y_n = \displaystyle\int_0^1 \frac{(f(x))^{n+1}}{(g(x))^n} dx$, for every integer $n \geq 0$. Show that $\{y_n\}_{n\geq 0}$ is an increasing sequence.

Solution. To start with, note that $y_0 = \int_0^1 f = \int_0^1 g$, and $y_1 = \int_0^1 f^2/g$. How to show $y_0 \leq y_1$? Well, the Cauchy-Schwarz inequality gives

$$\left(\int_0^1 \frac{f^2}{g}\right)\left(\int_0^1 g\right) \geq \left(\int_0^1 f\right)^2 \implies y_1 y_0 \geq y_0^2 \implies y_1 \geq y_0.$$

Let's proceed by strong induction. Suppose that $y_k \leq y_{k+1}$ holds for all $k \leq n-1$. How can we show that $y_n \leq y_{n+1}$. Cauchy-Schwarz inequality gives

$$\left(\int_0^1 \frac{f^{n+2}}{g^{n+1}}\right)\left(\int_0^1 \frac{f^n}{g^{n-1}}\right) \geq \left(\int_0^1 \frac{f^{n+1}}{g^n}\right)^2$$

which tells us that $y_{n+1}y_{n-1} \geq y_n^2$. Hence $y_{n+1}/y_n \geq y_n/y_{n-1}$ and $y_n/y_{n-1} \geq 1$ holds by induction hypothesis. This completes the induction and hence the proof. □

5. Suppose that f is integrable on $[a,b]$. Define

$$F(x) = \int_a^x f(t)\ dt, \text{ for } a \leq x \leq b.$$

Then, (i) F is continuous on $[a,b]$, and (ii) if f is continuous at $c \in [a,b]$, then F will be differentiable at c, with $F'(c) = f(c)$.

Solution. To be discussed in the next class. ⊠

6. If f is differentiable on $[a,b]$ such that f' is continuous on $[a,b]$, then

$$\int_a^b f'(x)\ dx = f(b) - f(a).$$

Solution. Define $F(x) = \int_a^x f'(t)\ dt$ for $t \in [a,b]$. Then by the previous exercise, we can say that F is differentiable on $[a,b]$, with $F'(t) = f'(t)$ for every $t \in [a,b]$. In other words, the function $g = F - f$ will be a differentiable function having derivative equal to 0 on entire $[a,b]$, which implies that g must be a constant function (this may be justified using MVT). Thus, $F(x) - f(x) = c$ for every $x \in [a,b]$. Putting $x = a$, and using $F(a) = 0$ (from its definition), we get that $c = -f(a)$. Therefore, $\int_a^b f'(t)\ dt = F(b) = f(b) + c = f(b) - f(a)$, which completes the proof. □

7. If f is continuous on $[a,b]$, show that $\displaystyle\int_a^b f(t)\ dt = f(c)(b-a)$ must hold for some $c \in (a,b)$.

   Solution. Easy. Just apply Rolle's theorem or the Mean Value Theorem on the function

$$F(x) = \int_a^x f(t)dt,\ \ x \in [a,b]$$

   which is differentiable here since f is continuous. $\square$

8. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_0^1 f(x)\ dx = 1$. Show that there exists a point $c \in (0,1)$ such that $f(c) = 3c^2$.

   Solution. Define

$$g(x) = \int_0^x f(t)\ dt - x^3,\ \ x \in [0,1].$$

   Note that $g(1) = g(0) = 0$, and invoking the FTC we can say that g is continuous on $[0,1]$ and differentiable on $(0,1)$. Hence we can apply Rolle's theorem on g, which gives the desired conclusion. $\square$

9. Prove the following inequalities:

$$\frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx \leq \frac{2\pi^2}{9}.$$

   Solution. Using the fact that $f(x) = \sin x$ being an increasing function on $[0,\pi/2]$, for $\pi/6 < x < \pi/2$ we have $1/2 < \sin x < 1$. Hence

$$\int_{\pi/6}^{\pi/2} \frac{x}{1} dx \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx \leq \int_{\pi/6}^{\pi/2} \frac{x}{1/2} dx.$$

   Observing that $\displaystyle\int_{\pi/6}^{\pi/2} x dx = \frac{\pi^2}{9}$, we are done! $\square$
