Introductory Problems on Derivatives

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1. Find the derivatives (if they exist) of the following functions:

   (a) $f(x) = x|x|$, $x \in \mathbb{R}$

   (b) $f(x) = \log|x|$, $x \in \mathbb{R}\backslash\{0\}$

   (c) $f(x) = \lfloor x \rfloor \sin^2(\pi x)$, $x \in \mathbb{R}$

   (d) $f(x) = (x - \lfloor x \rfloor)\sin^2(\pi x)$, $x \in \mathbb{R}$

2. Study the differentiability of the following functions

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \qquad g(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

3. Study differentiability of the following function

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \leq 1, \\ \dfrac{\pi}{4}\dfrac{|x|}{x} + \dfrac{x-1}{2} & \text{if } |x| > 1. \end{cases}$$

4. Assume that f and g are differentiable at a. Find the following limits

   (a) $\lim\limits_{x \to a} \dfrac{xf(a) - af(x)}{x - a}$,

   (b) $\lim\limits_{x \to a} \dfrac{f(x)g(a) - f(a)g(x)}{x - a}$,

   (c) $\lim\limits_{n \to \infty} n\left(f\left(a + \dfrac{1}{n}\right) + f\left(a + \dfrac{2}{n}\right) + \cdots + f\left(a + \dfrac{k}{n}\right) - kf(a)\right)$.

5. Assume that $f(0) = 0$ and that $f(x)$ is differentiable at $x = 0$. Find the value of

$$\lim_{x \to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + f\left(\frac{x}{3}\right) + \cdots + f\left(\frac{x}{2019}\right)\right).$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function continuous at 0, such that $\lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = \ell$. Prove that $f(x)$ must be differentiable at $x = 0$ and also find $f'(0)$.

7. Suppose that $a_1, a_2, \ldots, a_n$ are n real numbers such that

$$|a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx| \leq |\sin x|$$

   holds for every $x \in \mathbb{R}$. Prove that, $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

8. Let f and g be two functions such that $g(f(x))$ is well-defined (in an open neighborhood of a). Suppose that $f'(a)$ and $g'(f(a))$ both exist. The goal of this exercise is to prove the *chain rule*: $(g \circ f)'(a) = g'(f(a))f'(a)$.

   (a) Assuming that $f(x) \neq f(a)$ holds for all $x \in (a-\epsilon, a+\epsilon)\backslash\{a\}$ (for some $\epsilon > 0$), prove that $(g \circ f)'(a) = g'(f(a))f'(a)$.

   (b) Define
   $$Q(x) = \begin{cases} \dfrac{g(f(x)) - g(f(a))}{f(x) - f(a)} & \text{if } f(x) \neq f(a), \\ g'(f(a)) & \text{if } f(x) = f(a). \end{cases}$$
   Show that
   $$\frac{g(f(x)) - g(f(a))}{x - a} = Q(x) \cdot \frac{f(x) - f(a)}{x - a}$$
   holds for every $x \neq a$, regardless of whether $f(x) = f(a)$ or not.

   (c) Show that $\lim_{x \to a} Q(x) = g'(f(a))$.

   (d) Hence complete the proof of $(g \circ f)'(a) = g'(f(a))f'(a)$.

9. Give an example where $g'(f(a))$ does not exist, but $(g \circ f)'(a)$ exists.

10. Suppose that $g \circ f$ is differentiable at a, g is differentiable at $f(a)$, with $g'(f(a)) \neq 0$, and f is continuous at a. Then show that f must be differentiable at a and
   $$f'(a) = \frac{(g \circ f)'(a)}{g'(f(a))}.$$

11. (*Inverse Function Theorem*) Let $g : C \to D$ be an invertible function, with inverse $g^{-1} : D \to C$. Suppose that $c \in C$, and $d \in D$ are such that $d = g(c)$. If g is differentiable at c, g^{-1} is continuous at d, and $g'(c) \neq 0$, then show that g^{-1} must be differentiable at d and $(g^{-1})'(d) = 1/g'(c)$.

12. Show that the conclusion of the last problem does not hold if $g'(c) = 0$.

13. Calculate the limit $\displaystyle\lim_{n \to \infty} \left(\frac{f\left(a + \frac{1}{n}\right)}{f(a)} \right)^n$.

14. Determine, with proof, the value of $\displaystyle\lim_{n \to \infty} \tan^n \left(\frac{\pi}{4} + \frac{1}{n} \right)$.

15. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that,
   $$\lim_{x \to 0} \left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n} \right)^{1/x} = \sqrt[n]{a_1 a_2 \cdots a_n}.$$

16. The goal of this exercise is to show that $\frac{d}{dx}(x^r)$ is rx^{r-1} for any $r \in \mathbb{Q}$.

    (a) Show (from definition) that the derivative of x^n is nx^{n-1} where $n \in \mathbb{Z}$.

    (b) For any $n \in \mathbb{N}$, find the derivative of $x^{1/n}$. (Note, the function $x^{1/n}$ is usually restricted to positive values of x only, in order to avoid things like $(-1)^{1/4}$.)

    (c) Show that the derivative of x^r is rx^{r-1}, for any rational number r.

17. Define $f(x) = x^{\sqrt{3}}$ for $x > 0$. Is it true that $f'(x) = \sqrt{3}x^{\sqrt{3}-1}$? If yes, can you give a complete proof of this?

18. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$. Prove that

$$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}$$

    exists and equals $f'(c)$. Give an example of a function where this limit exists, but the function is not differentiable at $x = c$.

19. Let f be differentiable at a and let $\{x_n\}$ and $\{z_n\}$ be two sequences converging to a such that $x_n < a < z_n$ for every $n \geq 1$. Prove that,

$$\lim_{n \to \infty} \frac{f(z_n) - f(x_n)}{z_n - x_n} = f'(a).$$

    (Hint: Think intuitively; what does the above quotient represent?)

20. Let f be differentiable at a and let $\{x_n\}$ and $\{z_n\}$ be two sequences converging to a such that $x_n \neq a$, $z_n \neq a$ and $x_n \neq z_n$ for every $n \geq 1$. Furthermore, assume that the limit

$$\lim_{n \to \infty} \frac{f(z_n) - f(x_n)}{z_n - x_n}$$

    exists (finitely). Is it necessary that the above limit equals $f'(a)$?

21. Find a function which is differentiable everywhere except at exactly 5 points.

22. Define a function $f : (0, 2) \to \mathbb{R}$ as: $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 2x - 1 & \text{if } x \text{ is irrational} \end{cases}$. Determine all points where f is differentiable.

23. Can you give an example of a function which is differentiable at exactly 3 points?

24. Construct a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is not differentiable at the integers only (i.e., $f'(a)$ exists $\iff a \notin \mathbb{Z}$).

25. Construct a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at the integers only (i.e., $f'(a)$ exists $\iff a \in \mathbb{Z}$).

26. There exist functions that are continuous on $\mathbb{R}$ but not differentiable at any point! For example, Weierstrass function satisfies this property. Let $W(x)$ be any function (e.g., Weierstrass function) which is continuous everywhere and differentiable nowhere. Show that the function $\widetilde{W}(x) = (\sin \pi x)W(x)$ is an example of a continuous function which is differentiable exactly at the integers (i.e., $\widetilde{W}'(a)$ exists iff $a \in \mathbb{Z}$).

27. (*Leibnitz rule*) Show that for any $n \in \mathbb{N}$, the n-th derivative of $(fg)(x) = f(x)g(x)$ is given by

$$\frac{d^n}{dx^n}(fg)\Big|_{x=a} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k)}(a)$$

Of course the above equation makes sense only when $f(x), g(x)$ are both n-times differentiable. (Here $f^{(k)}(a)$ denotes the k-th derivative of f evaluated at $x = a$.)

28. Suppose that $xf(x) = \log x$ holds for all $x > 0$. Show that

$$f^{(n)}(1) = (-1)^{n-1}n!\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right),$$

where $f^{(n)}(a)$ denotes the n-th derivative of f evaluated at $x = a$.

29. Suppose that f, g are two differentiable functions such that $f(x) > g(x)$ holds for every $x \in \mathbb{R}$. Is it necessary that $f'(x) \geq g'(x)$?

30. Suppose that f is differentiable (on an interval I). Is it possible that

    (a) f is unbounded on I, but f' is bounded?

    (b) f is bounded on I, but f' is unbounded?

31. (*Lagrange's interpolation*) Let $P(x)$ be a polynomial of degree n with n distinct real roots $r_1, \ldots, r_n$ and let $Q(x)$ be a polynomial of degree at most $n - 1$. Prove that,

$$\frac{Q(x)}{P(x)} = \sum_{k=1}^{n} \frac{Q(r_k)}{P'(r_k)(x - r_k)}$$

holds for any $x \in \mathbb{R}\backslash\{r_1, \ldots, r_n\}$.